Singular Value Decomposition (SVD) applications

Mathematical Tools for Neuroscience (NEU 314)
Fall, 2021

lecture 9

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Notes:

1) No in-person class on Thursday (5/7)
2) Labs will be *optional* during midterms week (Oct 11-15)

Recap of SVD

Singular Value Decomposition (SVD)

$$A = USV^\top$$

$$U = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix}$$

left singular vectors

(orthogonal/unitary)

$$U^T U = U U^T = I$$

Singular Value Decomposition (SVD)

$$A = USV^\top$$

$$U = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix} \qquad S = \begin{bmatrix} s_1 & & & \\ & s_2 & & \\ & & \ddots & \\ & & & s_n \end{bmatrix}$$

left singular vectors (orthogonal/unitary)

$U^TU = UU^T = I$

singular values (all ≥ 0)

$s_1 \geq s_2 \geq \dots \geq s_n$ (by convention)

Singular Value Decomposition (SVD)

$$A = USV^\top$$

$$U = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix} \quad S = \begin{bmatrix} s_1 & & & \\ & s_2 & & \\ & & \ddots & \\ & & & s_n \end{bmatrix} \quad V^\top = \begin{bmatrix} - & v_1 & - \\ - & v_2 & - \\ & \vdots & \\ - & v_n & - \end{bmatrix}$$

left singular vectors (orthogonal/unitary)

$U^T U = U U^T = I$

singular values (all ≥ 0)

right singular vectors (orthogonal/unitary)

$V^T V = V V^T = I$

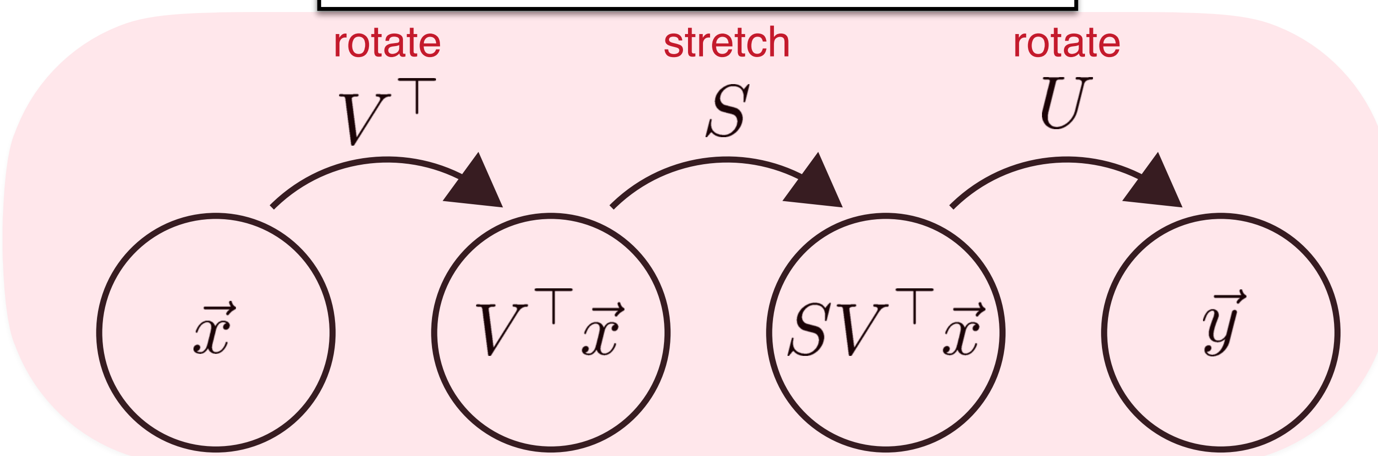
$s_1 \geq s_2 \geq \ldots \geq s_n$ (by convention)

conceptual picture

This means we can think of the linear map, $\vec{x} \longrightarrow A\vec{x}$
in terms of the following three steps:

1. rotation (multiplication by $V^\top$, which doesn't change vector length of $\vec{x}$).
2. stretching along the cardinal axes (where the $i'th$ component is stretched by s_i).
3. another rotation (multipication by U).

$$\vec{y} = A\vec{x} = (USV^\top)\,\vec{x}$$



inverse

$$A = USV^\top$$

$$\begin{bmatrix} \frac{1}{s_1} & & & \\ & \frac{1}{s_2} & & \\ & & \ddots & \\ & & & \frac{1}{s_n} \end{bmatrix}$$

inverse: $$A^{-1} = VS^{-1}U^\top$$

inverse

let's check:

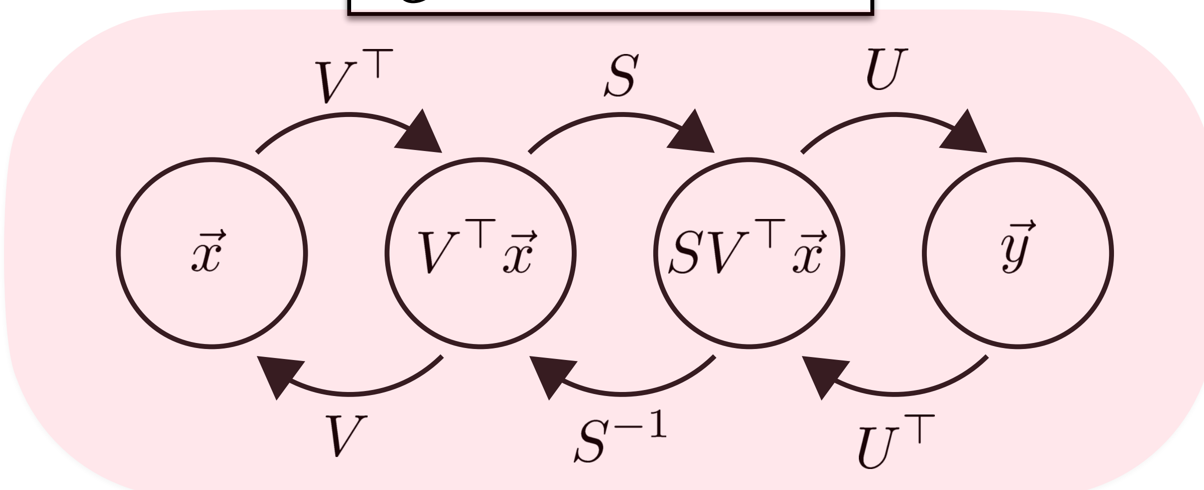
$$
\begin{aligned}
A^{-1}A &= (VS^{-1}U^\top)(USV^\top) \\
&= VS^{-1}(U^\top U)SV^\top \\
&= V(S^{-1}S)V^\top \\
&= VV^\top \\
&= I
\end{aligned}
$$

inverse: $A^{-1} = VS^{-1}U^\top$

mapping by A and its inverse

$$A = USV^\top$$

$$\vec{y} = A\vec{x}$$



$$A^{-1}\vec{y} = \vec{x}$$

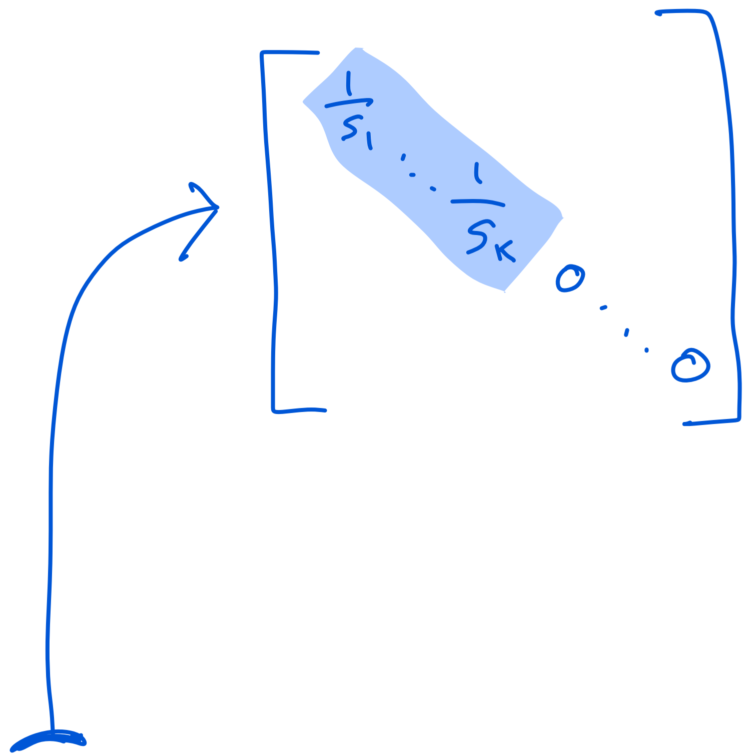
inverse: $$A^{-1} = VS^{-1}U^\top$$

Rank

rank(A) = # non-zero singular values

Pseudo-inverse

$$A = USV^\top$$



If A has k non-zero singular values,

pseudo-inverse: $A^\dagger = VS^\dagger U^\top$

Start Lecture 09:

Applications of SVD

- row/column/null spaces
- non-square matrices
- condition #
- eigenvectors / spectral theorem

Warmup Problems

Someone hands you the SVD of a matrix A: $A = USV^\top$

1. What is $A\overrightarrow{v}_1$?

where $\overrightarrow{v}_1$ = 1st right singular vector (i.e., the first row vector in V^T)

2. What is the SVD of A times its transpose?

$$AA^\top = \quad ?$$

3. What is the SVD of A-transpose times A?

$$A^\top A = \quad ?$$

Answers

Someone hands you the SVD of a matrix A: $A = USV^\top$

1. What is $A\overrightarrow{v}_1$?

where $\overrightarrow{v}_1$ = 1st right singular vector (i.e., the first row vector in V^T)

$$(USV^T)\overrightarrow{v}_1 = s_1\overrightarrow{u}_1$$

2. What is the SVD of A times its transpose?

- same L & R singular vectors
- singular values = s_i^2

Recall:
$(AB)^\top = B^\top A^\top$

$$AA^\top = (USV^\top)(VSU^\top)$$
$$= US^2U^\top$$

3. What is the SVD of A-transpose times A?

$$A^\top A = (VSU^\top)(USV^\top)$$
$$= VS^2V^\top$$

More Questions (group discussion)

How could you use SVD to:

1. determine whether a matrix is invertible?
2. find the rank of a matrix?
3. find an orthonormal basis for the row space?
4. find an orthonormal basis for the column space?
5. find an orthonormal basis for the *null space?*

Answers:

1. determine whether a matrix is invertible?

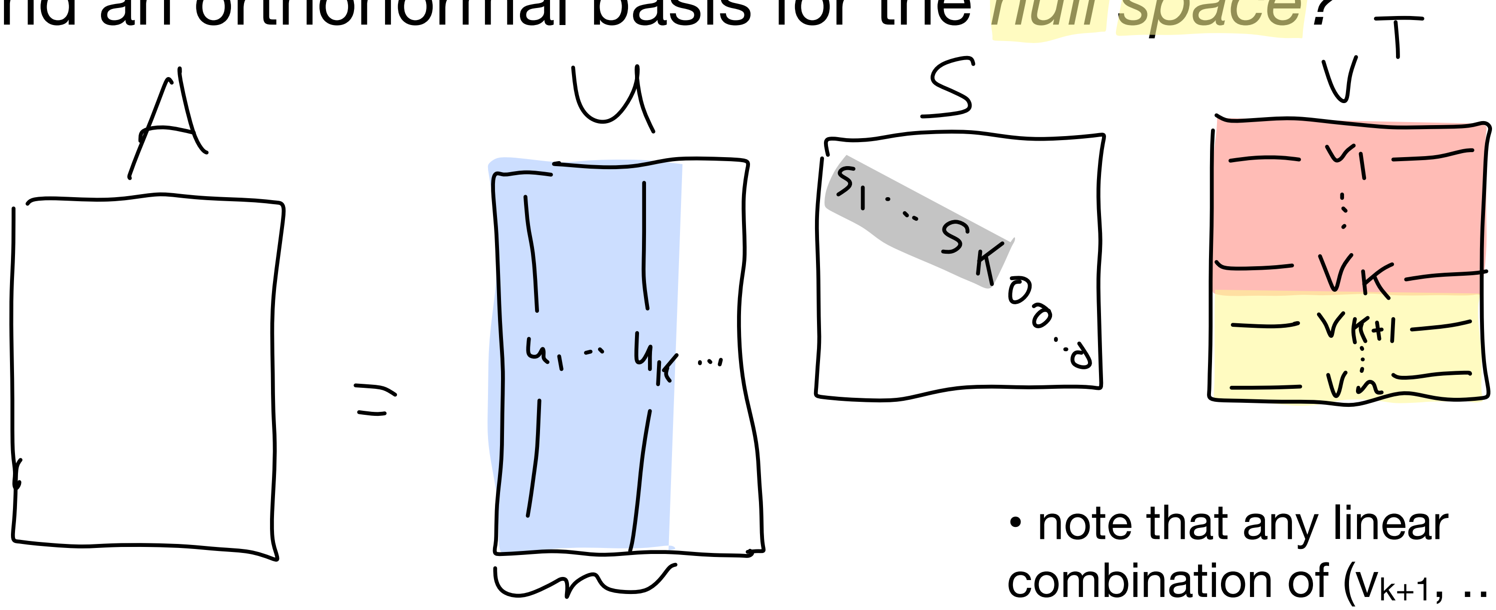
   Invertible if all singular values are > 0.

2. find the rank of a matrix?

   rank = # of non-zero singular values

Answers:

3. find an orthonormal basis for the row space?
4. find an orthonormal basis for the column space?
5. find an orthonormal basis for the *null space*?

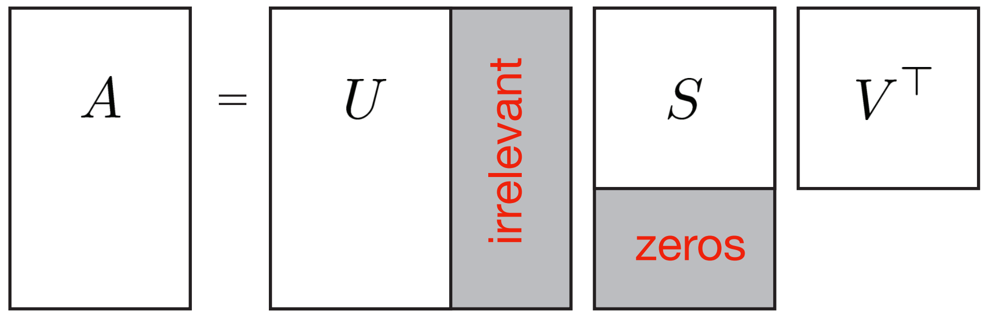


• note that any linear combination of (v_{k+1}, … v_n) has zero dot product with $v_1 \ldots v_k$, hence gives zero when multiplied by A (and is thus in null space!)

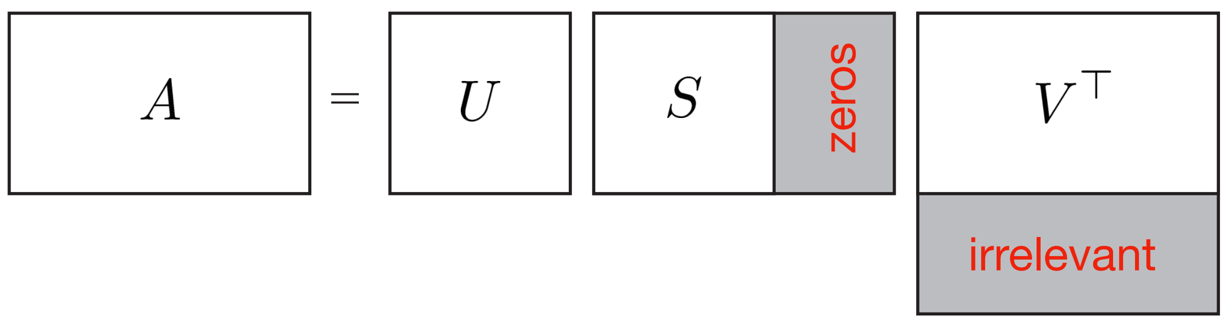
SVD of non-square matrices

SVD of non-square matrices

tall, skinny A:

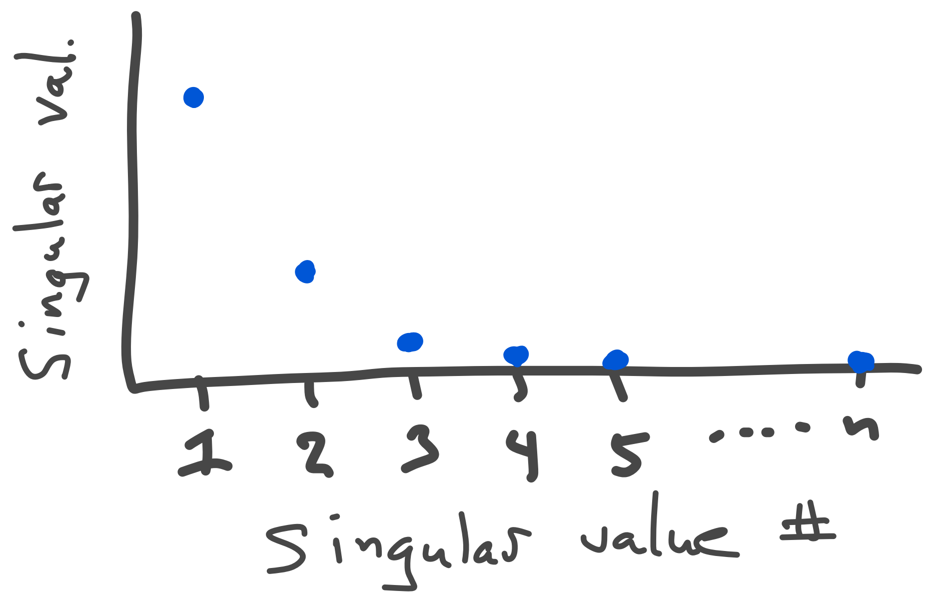


short, fat A:

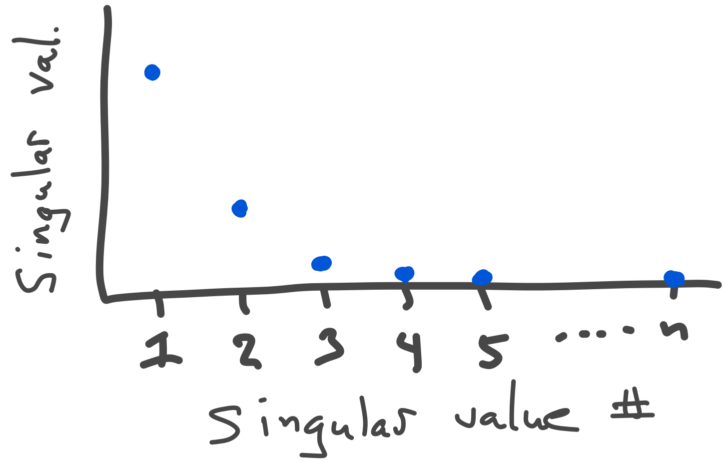


Condition number

true vs. practical non-invertibility



true vs. practical non-invertibility



condition number: $\frac{s_1}{s_n}$

largest sing. value

smallest sing. value

- matrix is not *practically* invertible if condition # too big (>10^{12})
- such a matrix called "ill-conditioned" or "singular"
- compute with: `numpy.linalg.cond`

eigenvectors / spectral theorem

eigenvectors

Q1: what is an eigenvector?

Q2: when is an eigenvector equal to a singular vector?

eigenvectors

Q1: what is an eigenvector?

- for a (square) matrix A, a vector $\vec{x}$ such that

$$A\vec{x} = \lambda\vec{x}$$

eigenvalue

eigenvector

that is, $A\vec{x}$ is a scaled version of $\vec{x}$.

positive semi-definite matrix

- matrix for which: $\vec{x}^\top A\vec{x} \geq 0$ for any vector $\vec{x}$

equivalent definition:

- a matrix for which all eigenvalues are ≥ 0

spectral theorem

If a matrix A is
- symmetric
- positive semi-definite

the singular value decomposition is also an eigen-decomposition:

$$A = USU^\top$$

matrix of (orthogonal) eigenvectors

eigenvalues along diagonal

- singular vectors = eigenvectors
- singular values = eigenvalues
- Note that left and right singular vectors are the same!

If a matrix A is

- symmetric
- positive semi-definite

} covariance matrices!

recall:

$$A^\top A = (VSU^\top)(USV^\top)$$
$$= VS^2V^\top$$

- V is matrix of orthogonal eigenvectors
- s_i^2 are eigenvalues

Summary

- condition number
- ill-conditioned / singular matrix
- eigenvectors & eigenvalues
- positive semi-definite matrices
- low-rank matrix approximation
- Frobenius norm (Euclidean norm for matrices)
- spectral theorem