Singular Value Decomposition (SVD)

Mathematical Tools for Neuroscience (NEU 314)
Fall, 2021

lecture 8

Jonathan Pillow

Quiz

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \quad v_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

(1) Is v_1 in the column space of A?

(2) Is v_1 in the row space of A?

(3) Is v_1 in the null space of A?

(4) What is the rank of A?

1/2 point each

(5a) Is this a linear function: $f(\vec{x}) = 3\vec{x} + 1$

(5b) Is this a linear function: $f(\vec{x}) = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \vec{x}$

Next up: the amazing SVD!

Daniela Witten @daniela_witten · Jul 5

Everybody knows that the #SVD is the #bestmatrixdecomposition !!!

#UDVt #minimumreconstructionerror #maximalvariance
#nonconvexbutstillsolvable
#nothanksQR
#seeyoulaterEigenDecomp
#blessed

Women in Statistics and Data Science @WomenInStat · Jul 21

So a lot of people think of the SVD as "just another matrix decomposition".

But today I will argue that if you are in statistics or data science, it is the #1 matrix decomposition, and likely the only one you will ever need.

And believe me: you are going to need it.

Women in Statistics and Data Science @WomenInStat · Jul 21

And please don't troll me with your comments about how you prefer the QR or LU decompositions.

I'm a working mom with 3 kids at home in the midst of a pandemic, I know you don't mean it, and I literally don't have time for this.

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Next up: the amazing SVD!

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quick recap:

1. What is an orthogonal matrix?

2. compute the transpose of the following two matrices

$$\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}^T \qquad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T$$

quick recap:

also known as "unitary" matrix

1. What is an orthogonal matrix?

- square matrix whose rows & columns are orthogonal unit vectors

$$B = \begin{pmatrix} | & | & & | \\ \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \\ | & | & & | \end{pmatrix} \qquad \begin{aligned} \vec{b}_i \cdot \vec{b}_i &= 1 \\ \vec{b}_i \cdot \vec{b}_j &= 0, i \neq j \end{aligned}$$

Properties:

$$BB^T = B^T B = I$$
$$B^{-1} = B^T$$

transpose is its inverse

$$||B\vec{v}|| = ||B^T\vec{v}|| = ||\vec{v}||$$

length-preserving

2. compute the transpose of the following two matrices

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2. compute the transpose of the following two matrices

$$\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

one weird fact about transposes & inverses

- **transpose of a product:**

$$(AB)^T = B^T A^T$$

(we can verify this with a simple example)

one weird fact about transposes & inverses

- **transpose of a product:**

$$(AB)^T = B^T A^T$$

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- **inverse of a product:**

$$(AB)^{-1} = B^{-1} A^{-1}$$

Singular Value Decomposition (SVD)

$$A = USV^\top$$

$$\begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix}$$

left singular vectors

(orthogonal/unitary)

$U^T U = U U^T = I$

Singular Value Decomposition (SVD)

$$A = USV^\top$$

$$U = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix} \qquad S = \begin{bmatrix} s_1 & & & \\ & s_2 & & \\ & & \ddots & \\ & & & s_n \end{bmatrix}$$

left singular vectors (orthogonal/unitary)

$U^T U = U U^T = I$

singular values (all ≥ 0)

$s_1 \geq s_2 \geq \ldots \geq s_n$ (by convention)

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left singular vectors (orthogonal/unitary)

$U^T U = U U^T = I$

singular values (all ≥ 0)

right singular vectors (orthogonal/unitary)

$V^T V = V V^T = I$

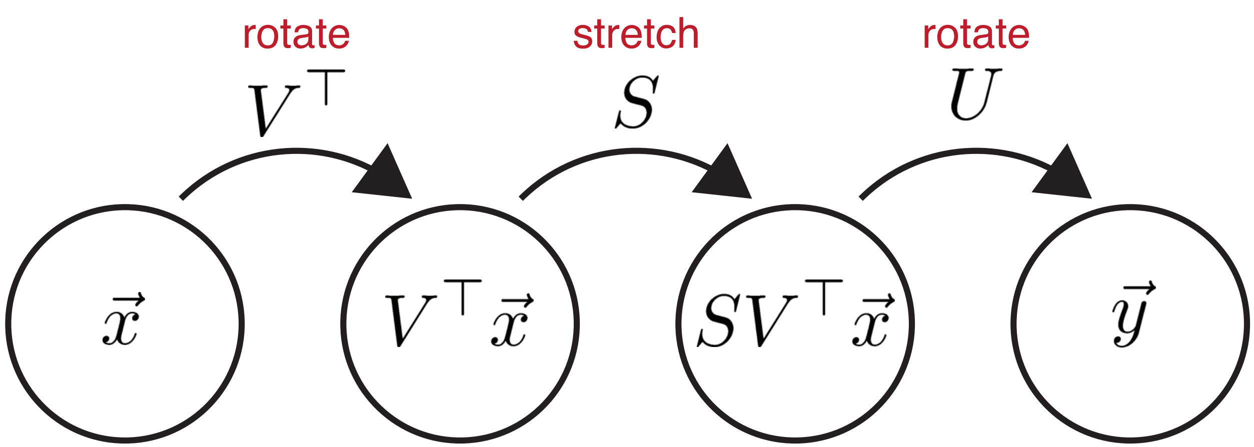
$s_1 \geq s_2 \geq \ldots \geq s_n$ (by convention)

conceptual picture

This means we can think of the linear map, $\vec{x} \longrightarrow A\vec{x}$
in terms of the following three steps:

1. rotation (multiplication by $V^\top$, which doesn't change vector length of $\vec{x}$).
2. stretching along the cardinal axes (where the $i'th$ component is stretched by s_i).
3. another rotation (multipication by U).

$$\vec{y} = A\vec{x} = (USV^\top)\,\vec{x}$$



inverse

$$A = USV^\top$$

$$\begin{bmatrix} \frac{1}{s_1} & & & \\ & \frac{1}{s_2} & & \\ & & \ddots & \\ & & & \frac{1}{s_n} \end{bmatrix}$$

inverse: $$A^{-1} = VS^{-1}U^\top$$

inverse

let's check:

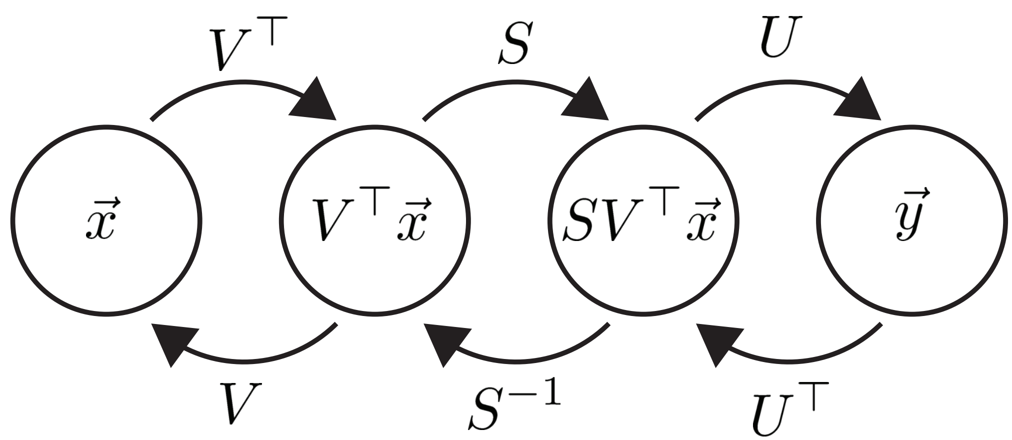
$$\begin{aligned} A^{-1}A &= (VS^{-1}U^\top)(USV^\top) \\ &= VS^{-1}(U^\top U)SV^\top \\ &= V(S^{-1}S)V^\top \\ &= VV^\top \\ &= I \end{aligned}$$

inverse: $$A^{-1} = VS^{-1}U^\top$$

can you check: AA^{-1}

mapping by A and its inverse

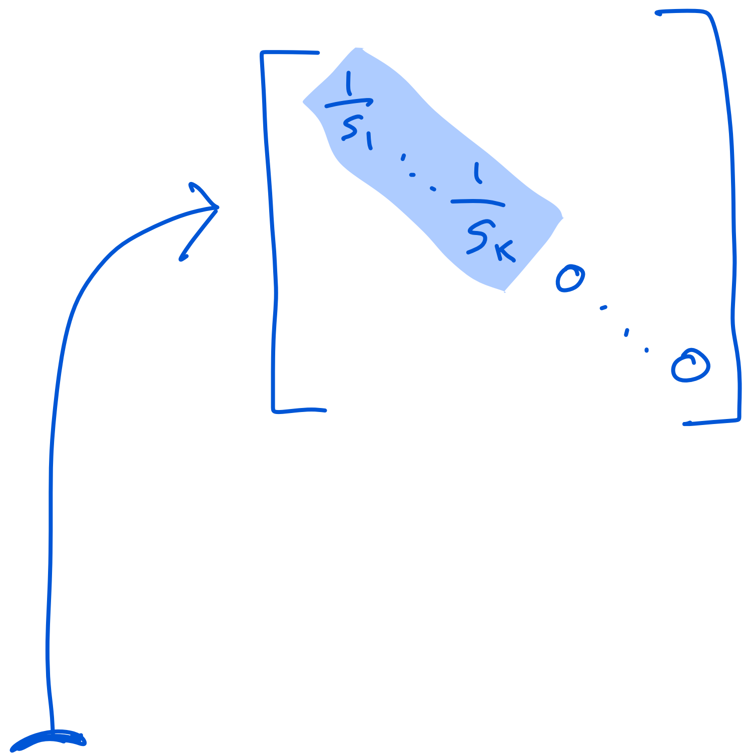
$$\vec{y} = A\vec{x}$$



$$A^{-1}\vec{y} = \vec{x}$$

Pseudo-inverse

$$A = USV^\top$$



If we have k non-zero singular values,

pseudo-inverse: $A^\dagger = VS^\dagger U^\top$

Rank

rank(A) = # non-zero singular values

if $s_1, \ldots, s_k > 0$
and $s_{k+1}, \ldots, s_n = 0$
rank = k

Summary

- transpose of a product
- inverse of a product
- singular value decomposition (SVD)
- inverse
- pseudo-inverse
- rank