Row / Column / Null Spaces

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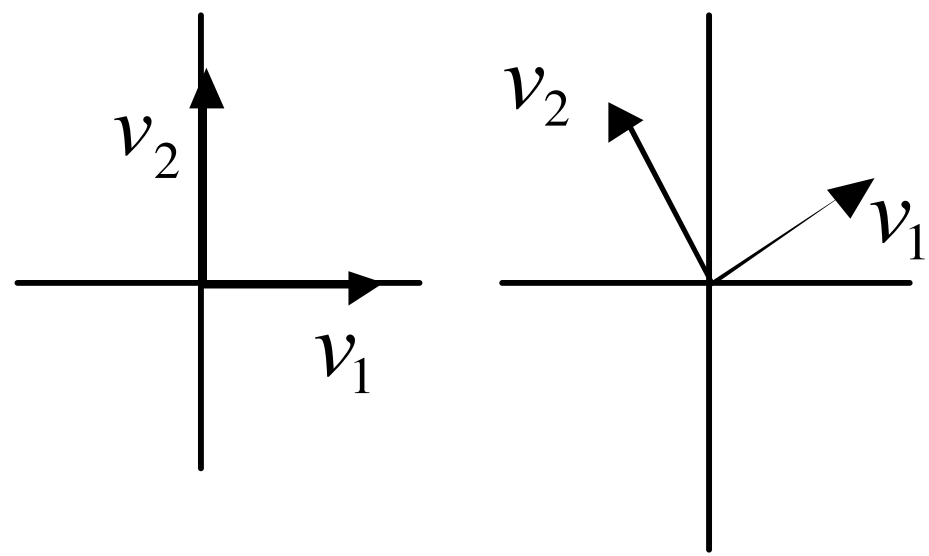
lecture 6

reminder: orthonormal basis

- basis composed of orthogonal unit vectors



- Two different orthonormal bases for the same vector space

Orthogonal matrix

- Square matrix whose columns (and rows) form an orthonormal basis (i.e., are orthogonal unit vectors)

$$B = \begin{pmatrix} | & | & & | \\ \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \\ | & | & & | \end{pmatrix}$$

$$\begin{aligned} \vec{b}_i \cdot \vec{b}_i &= 1 \\ \vec{b}_i \cdot \vec{b}_j &= 0, i \neq j \end{aligned}$$

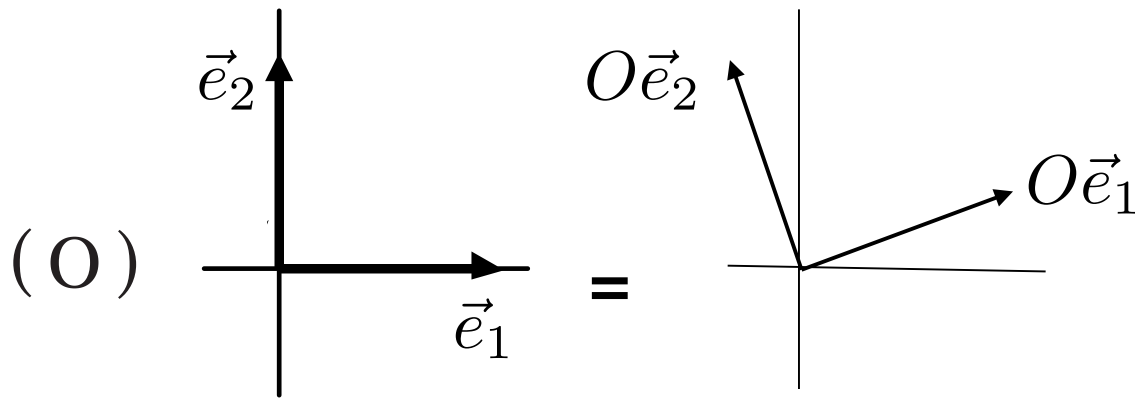
Properties:

$$BB^T = B^T B = I$$

$$B^{-1} = B^T$$

$$||B\vec{v}|| = ||B^T \vec{v}|| = ||\vec{v}||$$ length-preserving

- 2D example: rotation matrix



$$e.g.\ O = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

[flesh out on board]

Rank

- the **rank** of a matrix is equal to
    - # of linearly independent columns
    - # of linearly independent rows

(remarkably, these are always the same)

equivalent definition:

- the rank of a matrix is the *dimensionality* of the vector space spanned by its rows or its columns

for an *m x n* matrix *A*: $\text{rank}(A) \leq \min(m,n)$

(can't be greater than # of rows or # of columns)

Rank

Q: what is the rank of the outer product of two vectors?

$$\mathrm{rank}(\vec{a}\,\vec{b}^{\top})$$

One way to see this:

in an outer product,
every column is a scaled copy of $\vec{a}$

$$\vec{a}\,\vec{b}^{\top} = \begin{bmatrix} | & & | \\ \vec{a} \cdot b_1 & \cdots & \vec{a}\; b_n \\ | & & | \end{bmatrix}$$

and every row is a scaled copy of $\vec{b}$

$$\vec{a}\,\vec{b}^{\top} = \begin{bmatrix} a_1 \cdot & \text{———} & \vec{b} & \text{———} \\ & & \vdots & \\ a_m \cdot & \text{———} & \vec{b} & \text{———} \end{bmatrix}$$

Column space, Row Space & Null Space

- 3 vector spaces associated with any matrix

column space of a matrix W:

$n \times m$ matrix $$W = \begin{pmatrix} w_{11} & \cdots & w_{1m} \\ \vdots & & \vdots \\ w_{n1} & \cdots & w_{nm} \end{pmatrix}$$

vector space spanned by the *columns* of W

$$\begin{pmatrix} | & & | \\ c_1 & \cdots & c_m \\ | & & | \end{pmatrix}$$

- these vectors live in an n-dimensional space, so the column space is a subspace of $\mathbf{R}^n$

row space of a matrix W:

$n \times m$ matrix

$$W = \begin{pmatrix} w_{11} & \cdots & w_{1m} \\ \vdots & & \vdots \\ w_{n1} & \cdots & w_{nm} \end{pmatrix}$$

vector space spanned by the *rows* of W

$$\begin{pmatrix} \text{———} & r_1 & \text{———} \\ & \vdots & \\ \text{———} & r_n & \text{———} \end{pmatrix}$$

- these vectors live in an m-dimensional space, so the column space is a subspace of $\mathbf{R}^m$

null space of a matrix W:

$n \times m$ matrix

- the vector space consisting of all vectors that are orthogonal to the *rows* of W

$$\begin{pmatrix} \text{———} & r_1 & \text{———} \\ & \vdots & \\ \text{———} & r_n & \text{———} \end{pmatrix}$$

null space of a matrix W:

$n \times m$ matrix

$$\begin{pmatrix} \text{———} \; r_1 \; \text{———} \\ \vdots \\ \text{———} \; r_n \; \text{———} \end{pmatrix}$$

- the vector space consisting of all vectors that are orthogonal to the *rows* of W

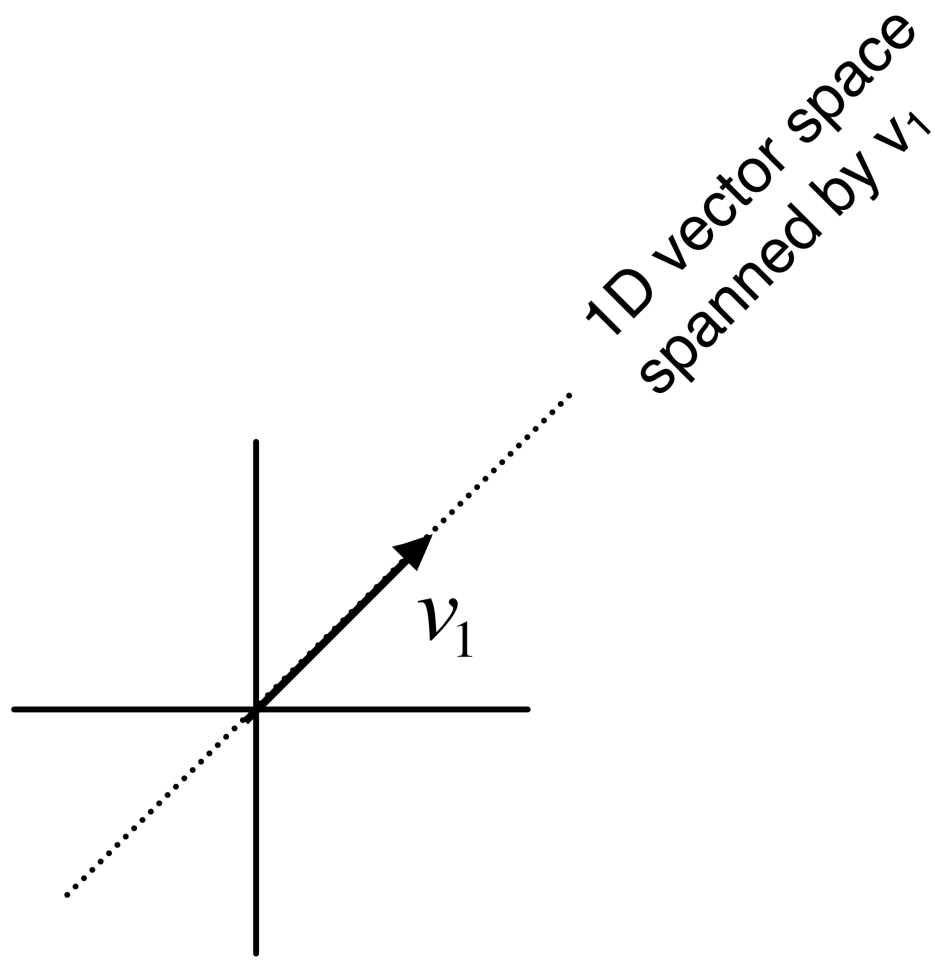
- equivalently: the null space of W is the vector space of all vectors x such that Wx = 0.

Q: is it obvious that the span of vectors orthogonal to to the rows of W is a vector space?

Answer: on board

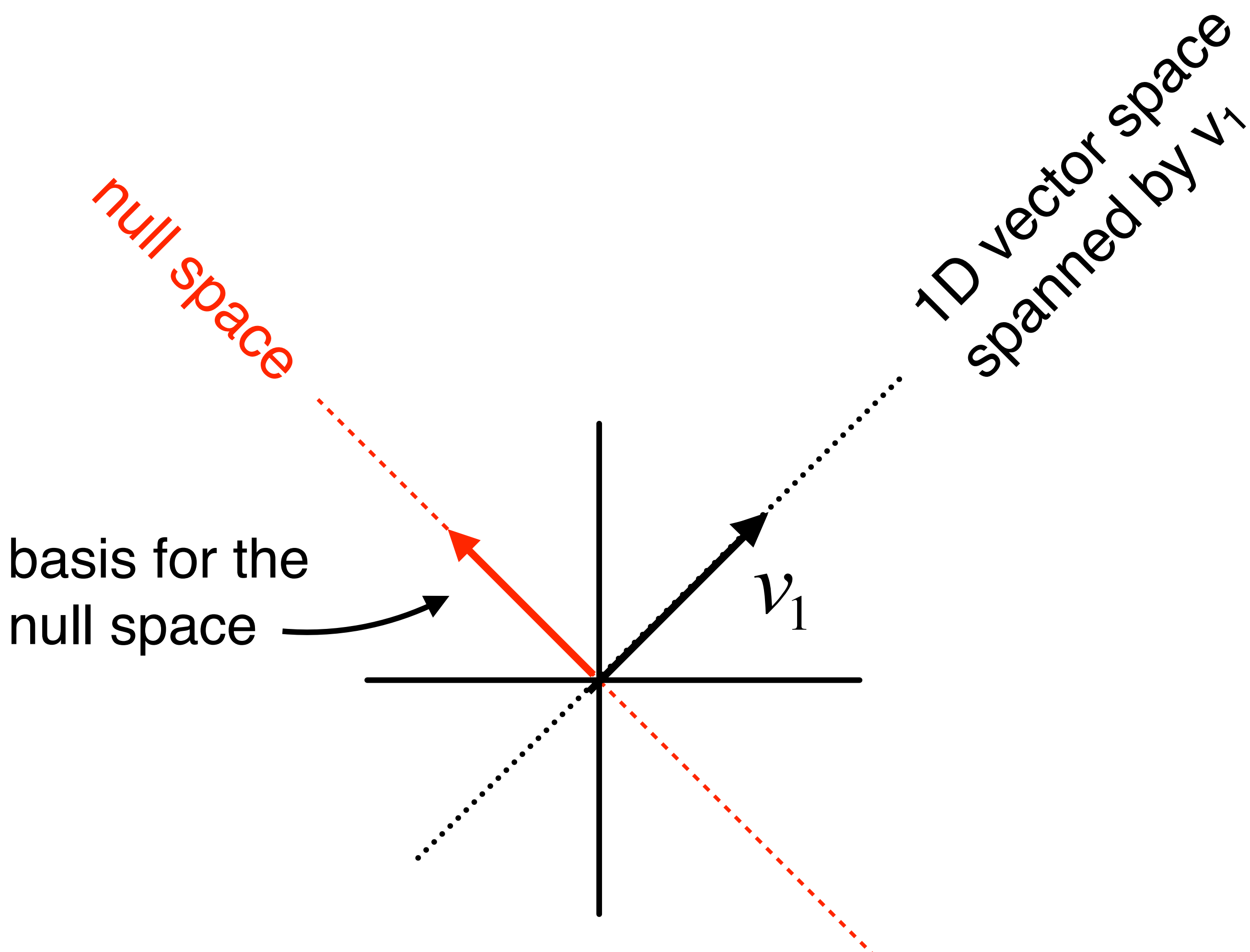
null space of a matrix W:

$W = (\text{———} \; v_1 \; \text{———})$



null space of a matrix W:

$$W = (\text{———} \; v_1 \; \text{———})$$



null space of a matrix W:

$n \times m$ matrix

$$\begin{pmatrix} \text{——— } r_1 \text{ ———} \\ \vdots \\ \text{——— } r_n \text{ ———} \end{pmatrix}$$

- the vector space consisting of all vectors that are orthogonal to the *rows* of W

- equivalently: the null space of W is the vector space of all vectors x such that Wx = 0.

- the null space is therefore entirely orthogonal to the row space of a matrix.

- Together, row space & null space make up all of $\mathbf{R}^m$

Linearity and Linear Systems

A **linear system** is a kind of mapping from vectors $\vec{x}$ to vectors $\vec{y}$

$$f(\vec{x}) \longrightarrow \vec{y}$$

such that the following two properties hold:

1. **homogeneity** ("scalar multiplication")

$$f(ax) = af(x)$$

2. **additivity**

$$f(\vec{x}_1 + \vec{x}_2) = f(\vec{x}_1) + f(\vec{x}_2)$$

Linearity and Linear Systems

(equivalent definition)

A **linear system** is a kind of mapping from vectors $\vec{x}$ to vectors $\vec{y}$

$$f(\vec{x}) \longrightarrow \vec{y}$$

that obeys the principle of **linear superposition**:

$$f(a\vec{x}_1 + b\vec{x}_2) = af(\vec{x}_1) + bf(\vec{x}_2)$$

for all scalars (*a*, *b*) and vectors ($\vec{x}_1$, $\vec{x}_2$)

in other words: "*the function of a linear combination is a linear combination of the function values*"

Linearity and Linear Systems

Question:

is the function $f(x) = ax + b$ a linear function?

why or why not?

Linearity and Linear Systems

FUN FACT:

Any linear system can be written as a matrix equation:

$$f(\vec{x}) = A\vec{x}$$

for some matrix *A.*

Excercise: let's check that Ax satistfies the definition of a linear function.

Summary

- orthogonal matrix
- rotation matrix
- rank
- column & row spaces
- null space
- linear systems
- linear superposition