More Matrix Stuff

(Multiplication, Outer product, Transpose, Identity, Inverse)

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lecture 5

Practice quiz

$$v_1 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

1. Are v_1 and v_2 linearly independent?
2. Are v_1 and v_2 orthogonal?
3. Compute the component v_1 in the direction of v_2
4. Compute the component of v_1 that is orthogonal to v_2
5. Compute v_1 times the matrix:

$$\begin{bmatrix} 5 & 3 & 2 & 6 \\ 2 & 6 & 1 & 1 \end{bmatrix}$$

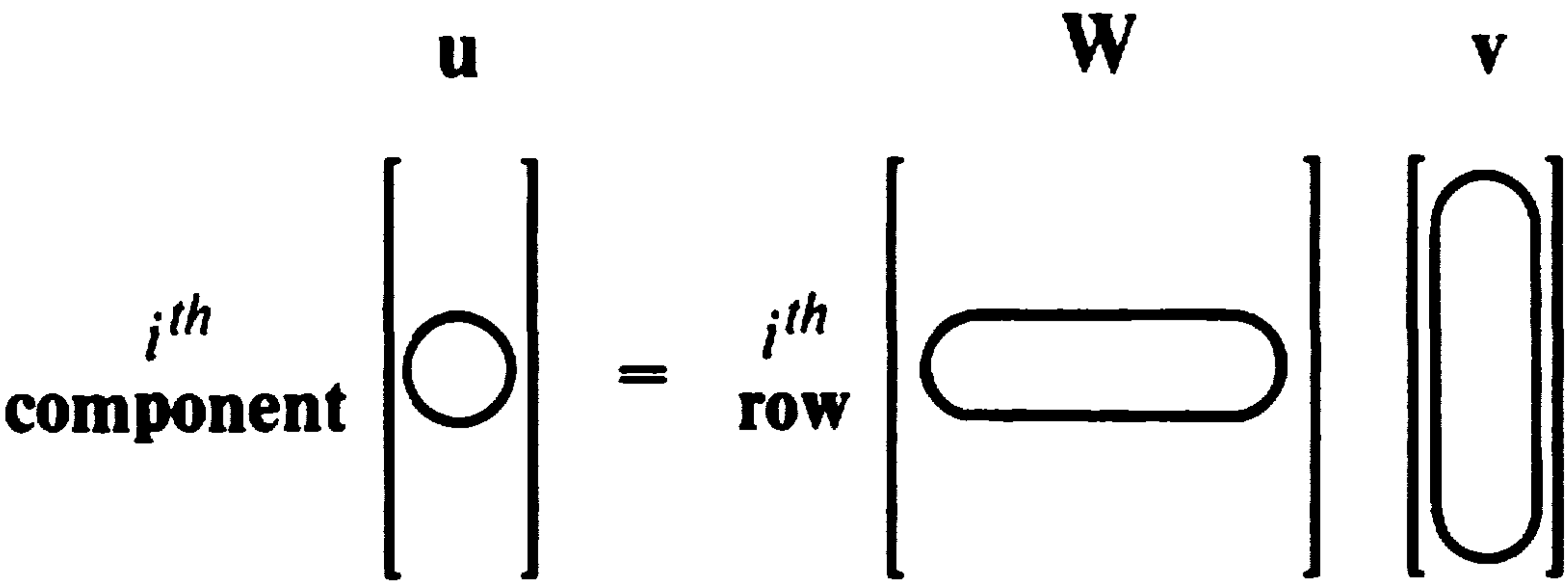
Quick Recap:

matrix-vector multiplication

matrix-vector multiplication

$$\vec{u} = W\vec{v}$$

One perspective: dot product with each row:



matrix-vector multiplication

another perspective: *linear combination of columns*

$$\underbrace{\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}}_{\vec{\mathbf{u}}} = \underbrace{\begin{bmatrix} | & & | \\ \vec{c}_1 & \cdots & \vec{c}_m \\ | & & | \end{bmatrix}}_{\mathbf{W}} \underbrace{\begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}}_{\vec{\mathbf{v}}}$$

$$= v_1 \cdot \begin{bmatrix} | \\ \vec{c}_1 \\ | \end{bmatrix} + v_2 \cdot \begin{bmatrix} | \\ \vec{c}_2 \\ | \end{bmatrix} + \ldots + v_m \cdot \begin{bmatrix} | \\ \vec{c}_m \\ | \end{bmatrix}$$

Matrix-Matrix Multiplication

$$AB = C$$

$$A \begin{bmatrix} | & | & & | \\ \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_k \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_k \\ | & | & & | \end{bmatrix}$$

A times each column of B

Matrix-Matrix Multiplication

$$AB = C$$

$$\begin{bmatrix} — & \vec{a}_1 & — \\ — & \vec{a}_2 & — \\ & \vdots & \\ — & \vec{a}_m & — \end{bmatrix} \begin{bmatrix} | & | & & | \\ \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_k \\ | & | & & | \end{bmatrix} = \begin{bmatrix} a_1 \cdot b_1 & a_1 \cdot b_2 & \cdots & a_1 \cdot b_k \\ a_2 \cdot b_1 & a_2 \cdot b_2 & & \\ \vdots & & \ddots & \\ a_m \cdot b_1 & & & a_m \cdot b_k \end{bmatrix}$$

dot products of rows of A with columns of B

Matrix-Matrix Multiplication

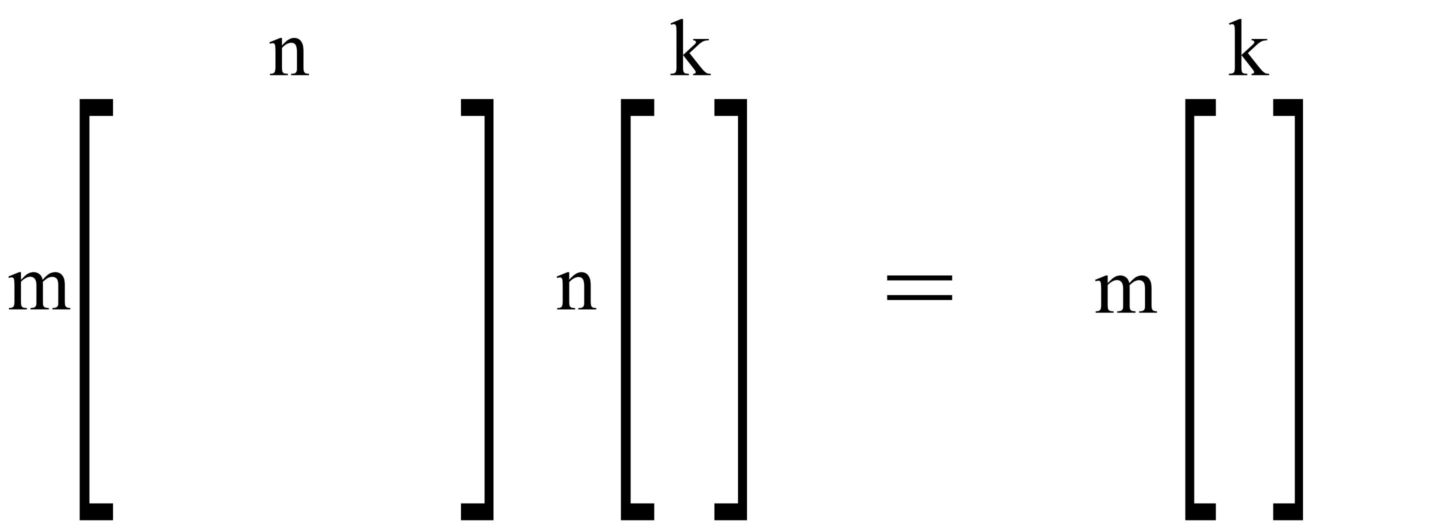
$$\underset{m \times n}{A}\underset{n \times k}{B} = \underset{m \times k}{C}$$

$$m\underbrace{\overset{n}{\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}}}\; n\overset{k}{\begin{bmatrix} & & \\ & B & \\ & & \end{bmatrix}} = m\overset{k}{\begin{bmatrix} & & \\ & C & \\ & & \end{bmatrix}}$$

- # columns of A must match # of rows of B

“Matrix Multiplication”

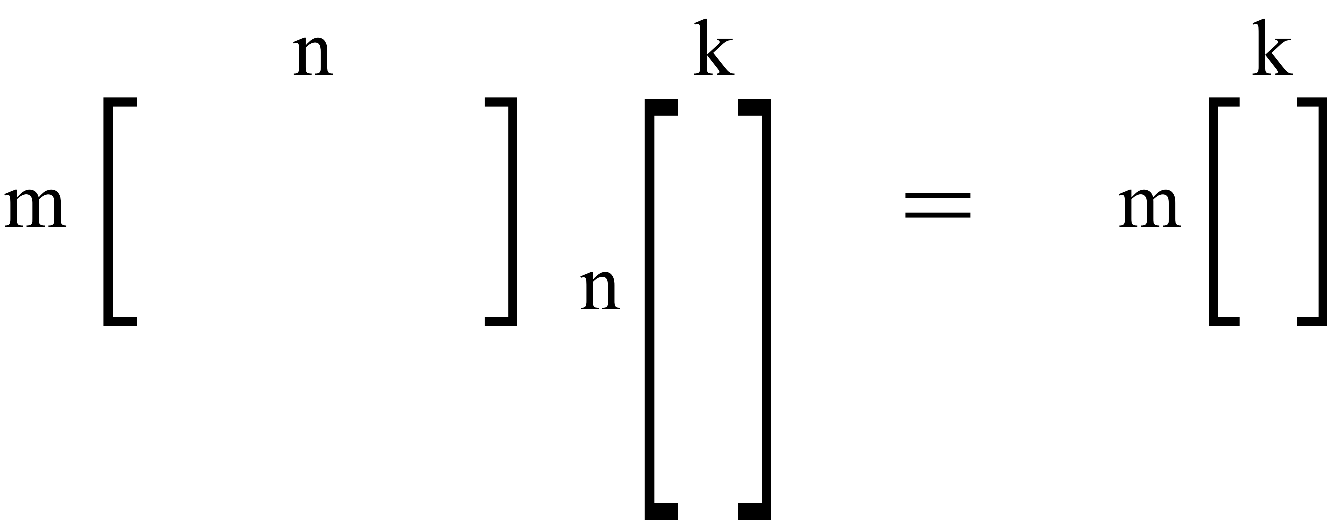
$$\underset{m \times n \quad n \times k}{AB} = \underset{m \times k}{C}$$



matrix times a vector: this is just a special case where k=1

Matrix Multiplication

$$\underset{m \times n}{A}\,\underset{n \times k}{B} = \underset{m \times k}{C}$$



short-fat matrix times a vector: short vector!

Matrix Multiplication

$$\underset{m \times n}{A}\ \underset{n \times k}{B} = \underset{m \times k}{C}$$

$$m\overset{n}{\begin{bmatrix} & & \end{bmatrix}}\ n\overset{k}{\begin{bmatrix} \\ \\ \end{bmatrix}} = m\overset{k}{\begin{bmatrix} & \end{bmatrix}}$$

short-fat matrix times tall-skinny matrix: short-skinny matrix

Matrix Multiplication

$$\vec{a}^\top \vec{b} = c$$

$1 \times n \qquad n \times 1$

$$\underset{1\times n}{\left[\quad\right]}\ \underset{n\times 1}{\left[\ \right]} = ?$$

what's this one called?

Matrix Multiplication

$$\underset{1 \times n}{A}\ \underset{n \times k}{B} = C$$

$$1 \overbrace{\begin{bmatrix} & \quad & \end{bmatrix}}^{n} \; n \overbrace{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}}^{k} = ?$$

what shape?

Matrix Multiplication

$$\vec{a}\,\vec{b}^{\top} = C$$

$$m \times 1 \qquad 1 \times k$$

$$\underset{m \times 1}{\begin{bmatrix} \\ \\ \\ \end{bmatrix}} \; \underset{1 \times k}{\begin{bmatrix} \quad & \quad & \quad \end{bmatrix}} = ?$$

called the **outer product**
of two vectors

Test yourself: matrix multiplication

1) $\begin{bmatrix} 5 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

2) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 11 & -2 \end{bmatrix}$

3) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 1 & 3 \\ 1 & 10 \end{bmatrix}$

4) $\begin{bmatrix} 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$

5) $\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \end{bmatrix}$ (outer product)

Matrix Multiplication

1. Associative

$$A(BC) = (AB)C$$

2. Not (in general) commutative

$$AB \neq BA$$

(One way in which matrix multiplication is unlike scalar multiplication!)

transpose A^T

- flipping around the diagonal
- rows become columns; columns become rows

$$\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

square matrix

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

non-square

the identity matrix I

- special matrix of all zeros except for 1 on the diagonal

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(3 x 3 identity matrix)

Q: what happens if we multiply a vector by the identity matrix?

inverse A^{-1}

- If A is a square matrix, its inverse A^{-1} satisfies:

$$AA^{-1} = A^{-1}A = I$$

- however: not all matrices are invertible (ie “have an inverse”) (for starters, A must be square…)

- in python: `np.linalg.inv(A)`

(Square) Matrix Equation

$$A\vec{x} = \vec{b}$$

assume square
and invertible

left-multiply both sides
by inverse of A:

$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$

$$I\vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

Summary

- Matrix multiplication
- outer product
- transpose
- identity matrix
- inverse of a matrix