Linear Combinations & Vector Spaces

Lecture 3

Thursday (9/9)

Math Tools for Neurosience (NEU 314)

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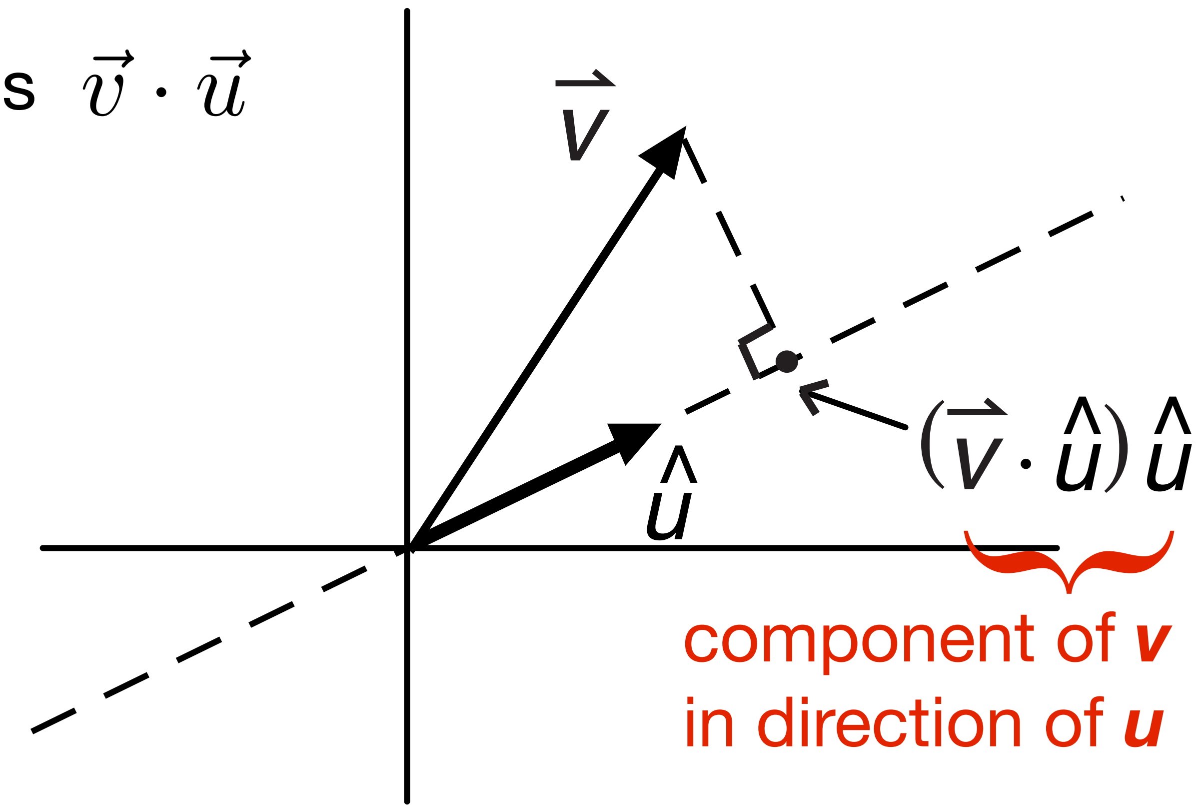
linear projection (review)

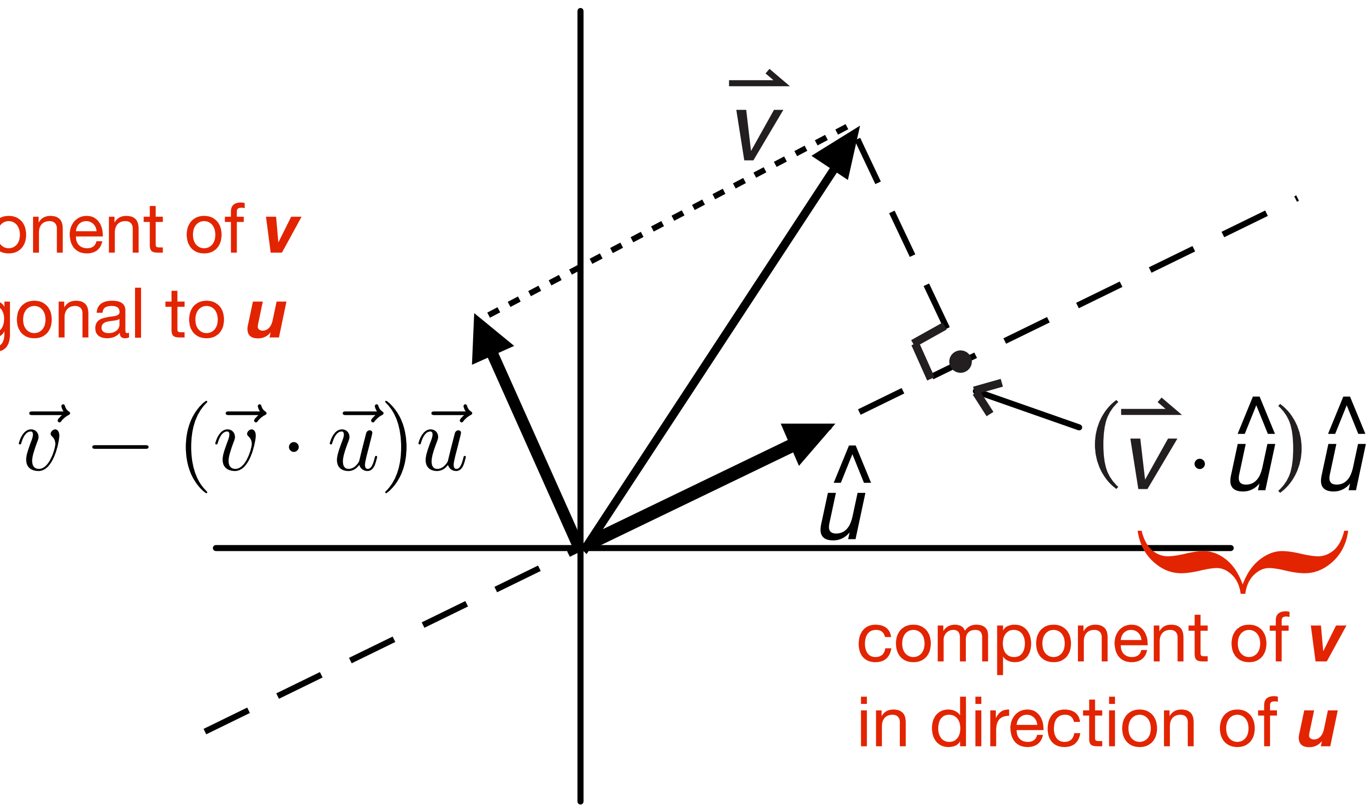
- intuitively, dropping a vector down onto a linear surface at a right angle
- if *u* is a unit vector, length of projection is $\vec{v} \cdot \vec{u}$



- for non-unit vector, length of projection = $\vec{v} \cdot \left(\frac{1}{||\vec{u}||} \vec{u} \right)$

orthogonality (review)

- two vectors are orthogonal (or "perpendicular") if their dot product is zero: $\vec{v} \cdot \vec{w} = 0$

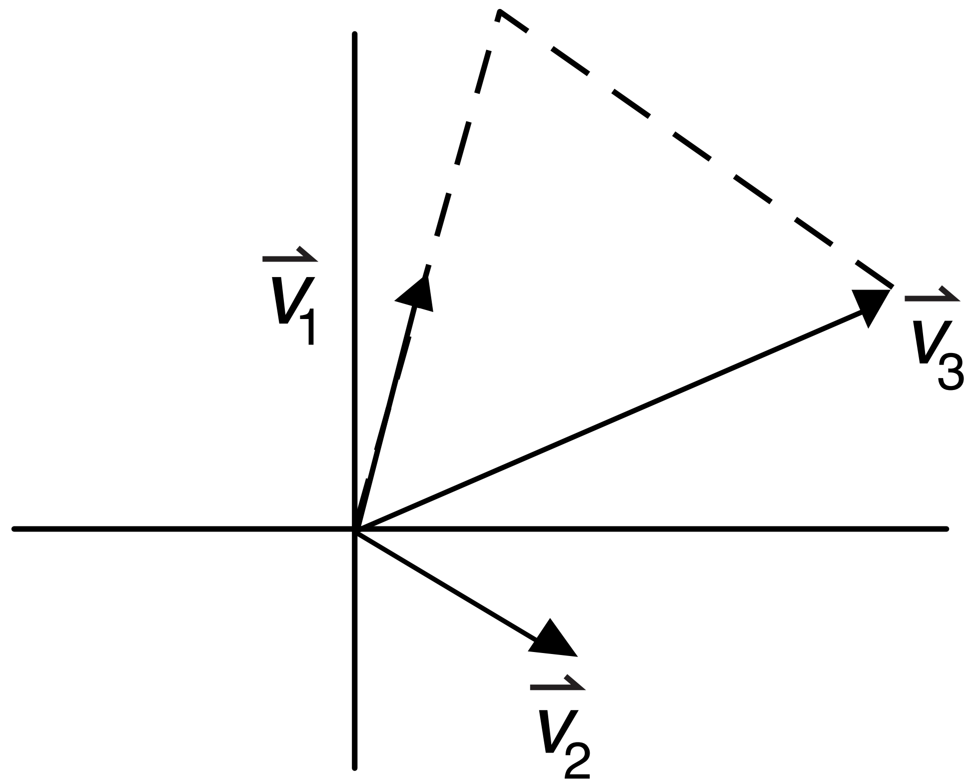


- Can decompose any vector into its component along u and its residual (orthogonal) component.

linear combination

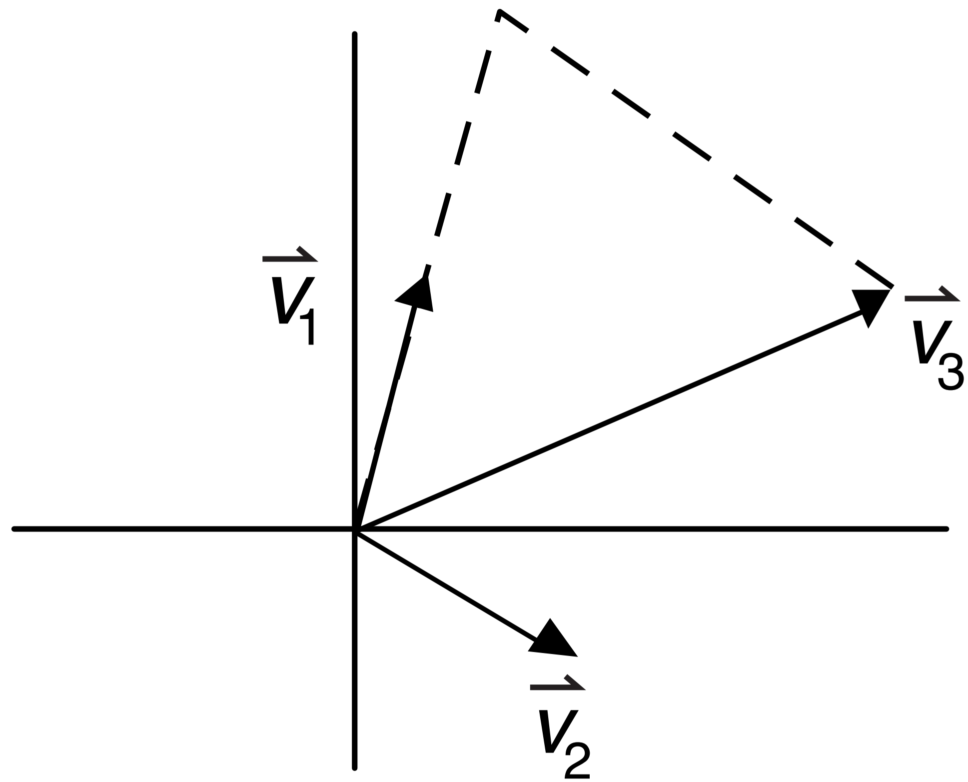
- scaling and summing applied to a group of vectors

$$a\vec{v}_1 + b\vec{v}_2 = \vec{v}_3$$



linear dependence & independence

- a group of vectors is ***linearly dependent*** if one can be written as a linear combination of the others

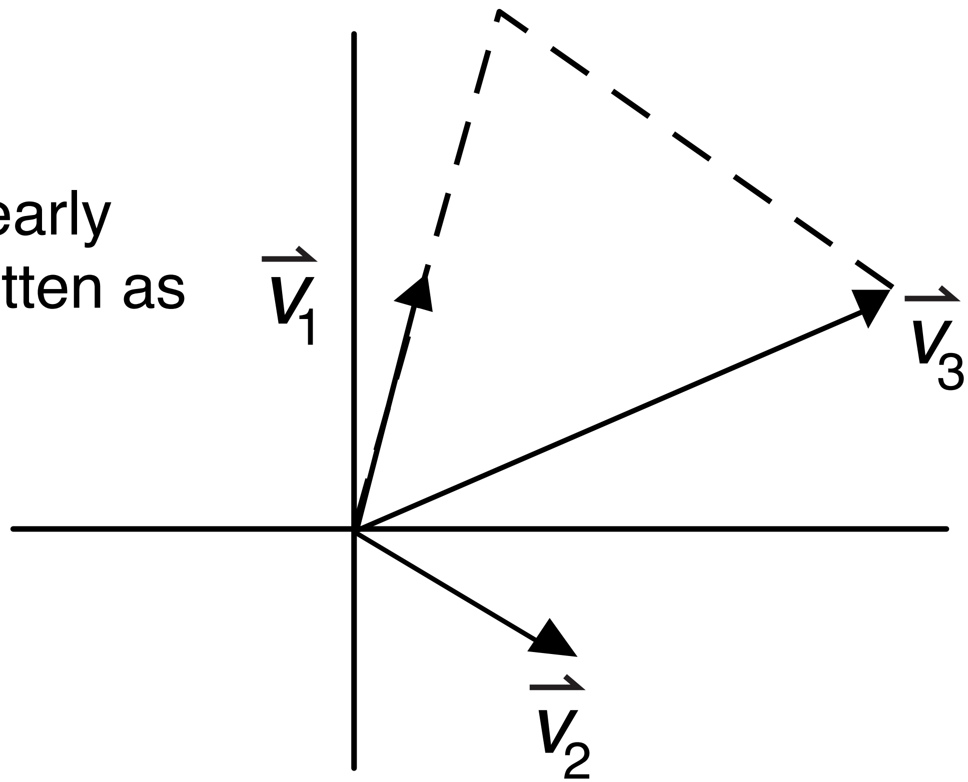


linear dependence & independence

- a group of vectors is ***linearly dependent*** if one can be written as a linear combination of the others

The vectors v_1, v_2, and v_3 and linearly dependent because v_3 can be written as a linear combination of v_1 and v_2:

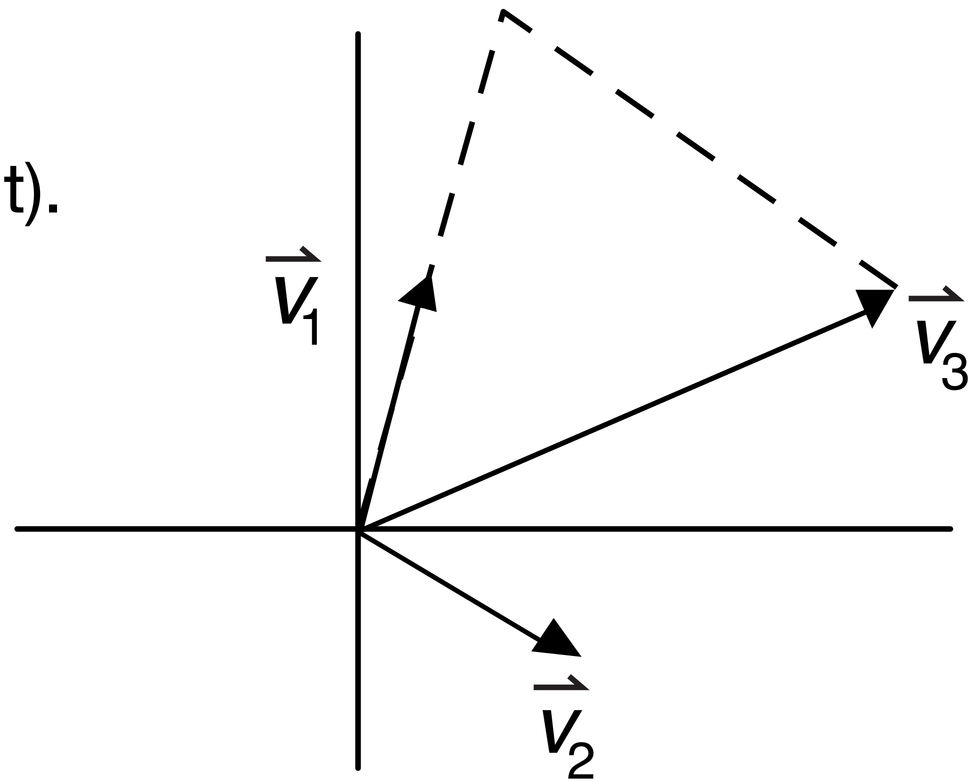
$$a\vec{v}_1 + b\vec{v}_2 = \vec{v}_3$$



linear dependence & independence

- a group of vectors is ***linearly independent*** if none of them can be written as a linear combination of the others

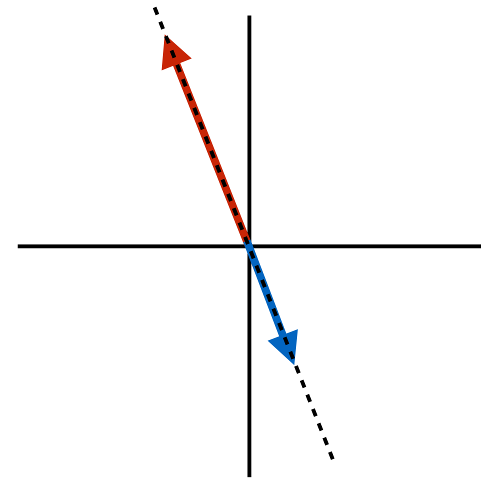
  (ie, is *not* linearly dependent).



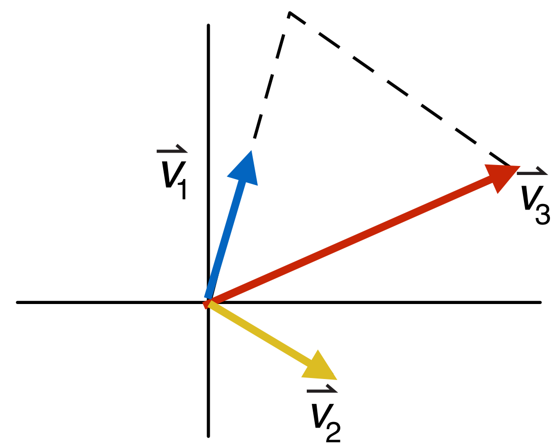
Question:

- if I have two vectors that are linearly *dependent*, what does that imply about them?

- 2 vectors in the same line: linearly dependent
- 2 vectors *not* in the same line: linearly independent



- 3 vectors in the same (2D) plane: linearly dependent
- 3 vectors *not* in the same plane: linearly independent



- 4 vectors in the same (3D) volume: linearly dependent
- 4 vectors *not* in the same 3D volume: linearly independent

Test yourself: linearly dependent or independent?

1) $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

2) $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

3) $\begin{bmatrix} -2 \\ 1 \\ -5 \end{bmatrix} \begin{bmatrix} 6 \\ -3 \\ 15 \end{bmatrix}$

4) $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} 7 \\ -1 \end{bmatrix}$

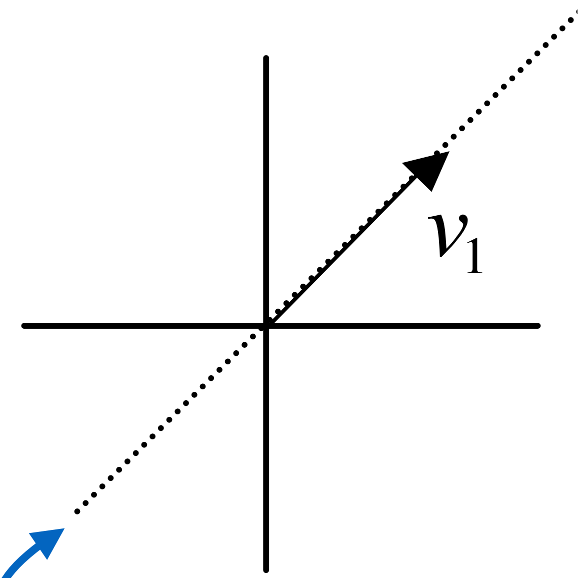
5) $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix}$

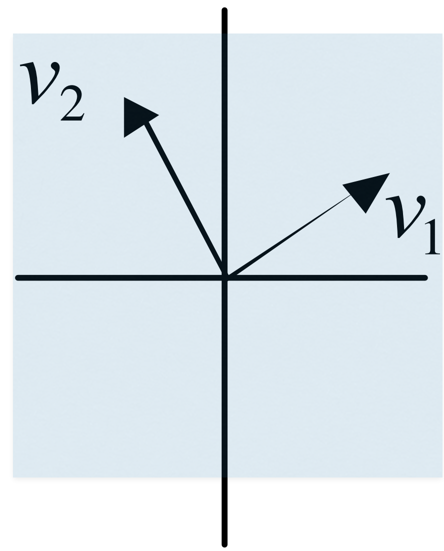
6) $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

(note: any set including the “0” vector is linearly dependent!)

from linear combinations: **vector space**

- set of all points that can be obtained by linear combinations of some set of **basis** vectors





1D vector space generated by scalar multiples of a single basis vector

2D vector space formed by all linear combinations ofbasis vectors v_1 and v_2

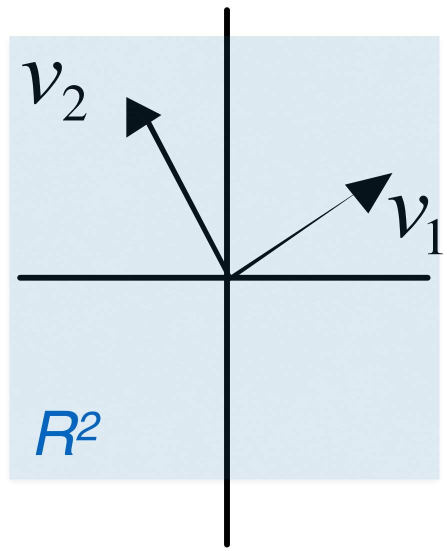
basis & span

- **basis** - set of vectors that can form (via linear combination) all points in a vector space
- **span** (verb) - to form (via linear combination) all points in a vector space.
- **span** (noun) - the vector space that results from all linear combinations of a set of vectors

So we would say:

- *V_1 and V_2 form a basis for the vector space R^2*
- *V_1 and V_2 span the vector space R^2*

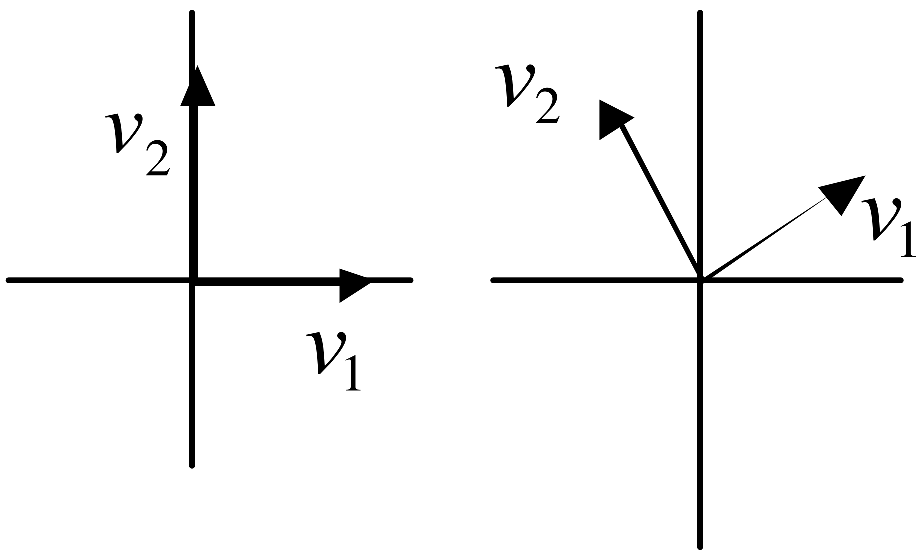
  or
- *The span of V_1 and V_2 is the vector space R^2*



R^2 = fancy name for “the 2D Euclidean plane”

orthonormal basis

- basis composed of orthogonal unit vectors



$V_1 \cdot V_1 = 1$

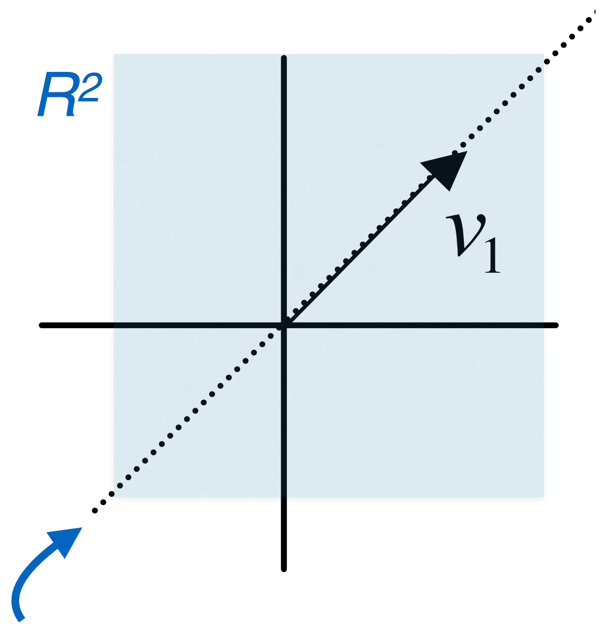
$V_2 \cdot V_2 = 1$

$V_1 \cdot V_2 = 0$

- Two different orthonormal bases for the same vector space

subspace

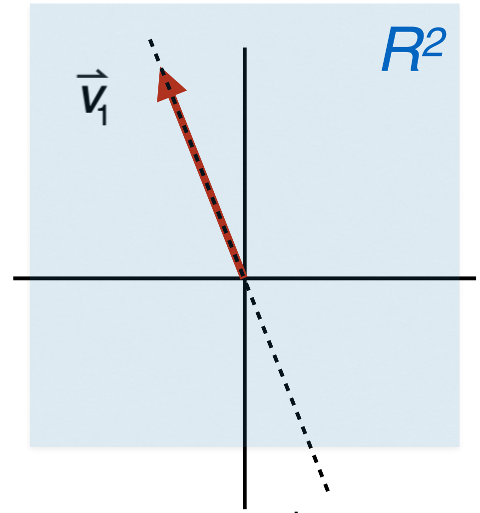
- **subspace** - a vector space contained inside another vector space



The 1D vector space spanned by v_1 is a subspace of the vector space R^2 (the full 2D plane).

so we'd say, for example:

- "The vector v_1 *spans* a 1D vector space".
- "The vector v_1 provides a *basis* for a 1D vector space".
- "That 1D vector space is a *subspace* of R^2, the 2D plane."



the vector space
(aka "the span of v_1")

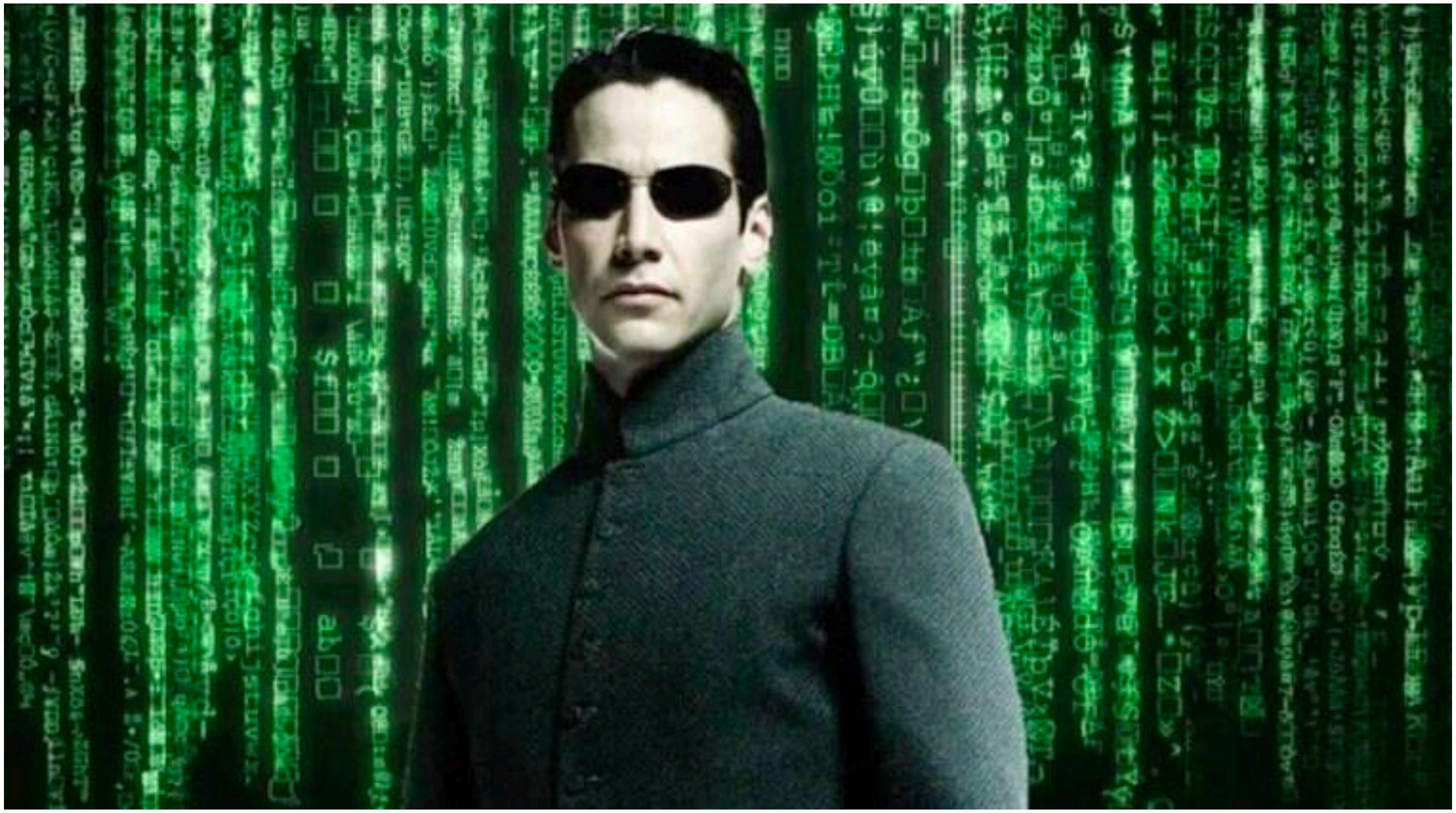
summary so far

- linear projection & orthogonality (review)

- linear combination
- linear independence / dependence
- vector space
- subspace
- basis
- span
- orthonormal basis

matrix

- a rectangular array of numbers



matrix

- a rectangular array of numbers

$$W = \begin{pmatrix} w_{11} & \cdots & w_{1m} \\ \vdots & & \vdots \\ w_{n1} & \cdots & w_{nm} \end{pmatrix}$$

$n \times m$ matrix

in python:

```
# make a 3 x 4 matrix
W = np.array([[1, 7, 3, 0], [2, -1, 2, -1], [1, 1, 1, 1]])
```

matrix

- a rectangular array of numbers

$$W = \begin{pmatrix} w_{11} & \cdots & w_{1m} \\ \vdots & & \vdots \\ w_{n1} & \cdots & w_{nm} \end{pmatrix}$$ $n \times m$ matrix

can think of it as:

m column vectors

$$\begin{pmatrix} | & & | \\ c_1 & \cdots & c_m \\ | & & | \end{pmatrix}$$

or

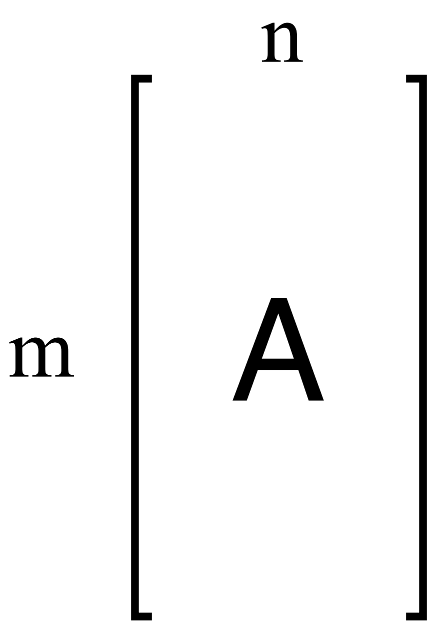
n row vectors

$$\begin{pmatrix} \text{———} \; r_1 \; \text{———} \\ \vdots \\ \text{———} \; r_n \; \text{———} \end{pmatrix}$$

we will often refer to a matrix as

"tall and skinny"

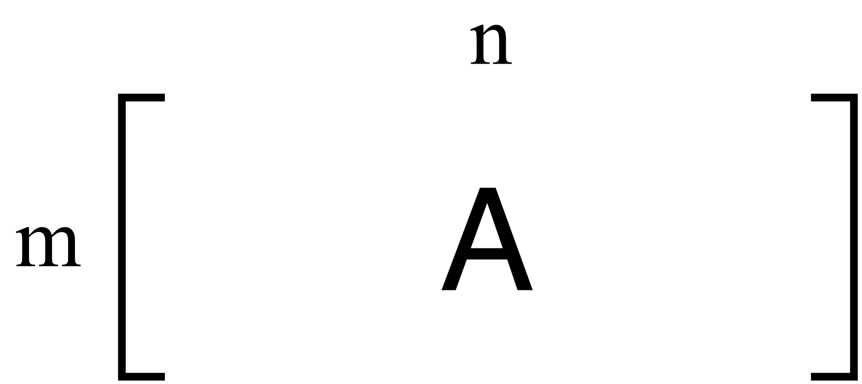
rows > # columns



or

"short and fat"

rows < # columns

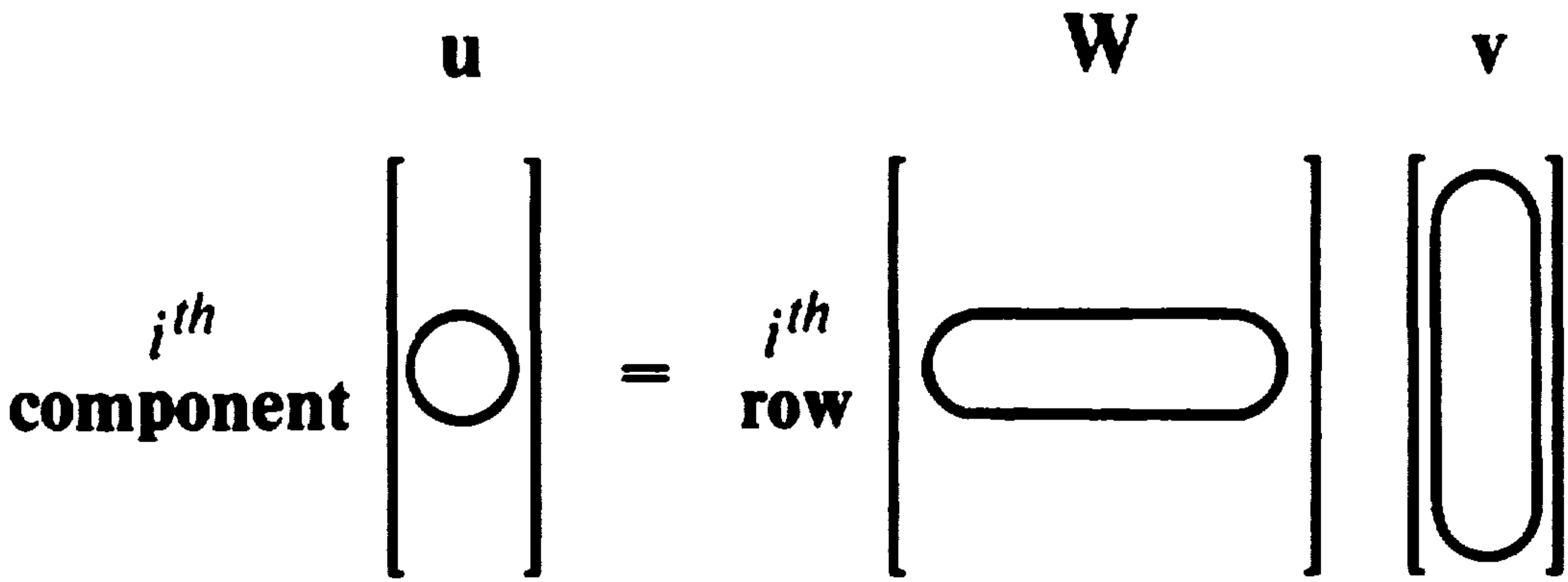


(but it's not a value judgment, obv!)

matrix-vector multiplication

$$\vec{u} = W\vec{v}$$

One perspective: dot product with each row:



matrix-vector multiplication

another perspective: *linear combination of columns*

$$\vec{\mathbf{u}} = \mathbf{W} \, \vec{\mathbf{v}}$$

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} | & & | \\ \vec{c_1} & \cdots & \vec{c_m} \\ | & & | \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$$

$$= v_1 \cdot \begin{matrix} | \\ \vec{c_1} \\ | \end{matrix} + v_2 \cdot \begin{matrix} | \\ \vec{c_2} \\ | \end{matrix} + \ldots + v_m \cdot \begin{matrix} | \\ \vec{c_m} \\ | \end{matrix}$$

Test yourself: matrix-vector multiplication

1) $\begin{bmatrix} 5 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 11 \end{bmatrix}$

3) $\begin{bmatrix} -2 & 6 \\ 1 & -3 \\ -5 & 15 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

4) $\begin{bmatrix} 1 & 3 & 7 \\ 2 & 4 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix}$

5) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

diagonal matrix

6) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Q1: What do you notice about the relationship between the size of the matrix and the size of the vector?

Q2: what does multiplying by a diagonal matrix do to a vector?

matrix-vector multiplication

you will never (or rarely) need to do this by hand!

in python:

```
# make a 3 x 4 matrix
W = np.array([[1, 7, 3, 0], [2, -1, 2, -1], [1, 1, 1, 1]])

# make a 4 x 1 matrix (ie, a vector)
v = np.array([[1], [2], [-3], [0]])


# Compute W times v (matrix-vector product)
u = W @ v
```

note special symbol '@' for matrix multiply!

Q: what size is u?

summary

- linear combination
- linear dependence / linear independence
- vector space
- subspace
- basis
- orthonormal basis
- span
- matrix-vector multiplication
- diagonal matrix (matrix with entries only along the diagonal)