Linear Algebra I: vectors and linear projection

NEU 314: Math Tools for Neurosience

Lecture 2 - Tuesday (9/7)

Jonathan Pillow

course website

http://pillowlab.princeton.edu/teaching/mathtools21fall/

1st quiz Thursday

(but no quiz next week)

Linear algebra

"Linear algebra has become as basic and as applicable as calculus, and fortunately it is easier."

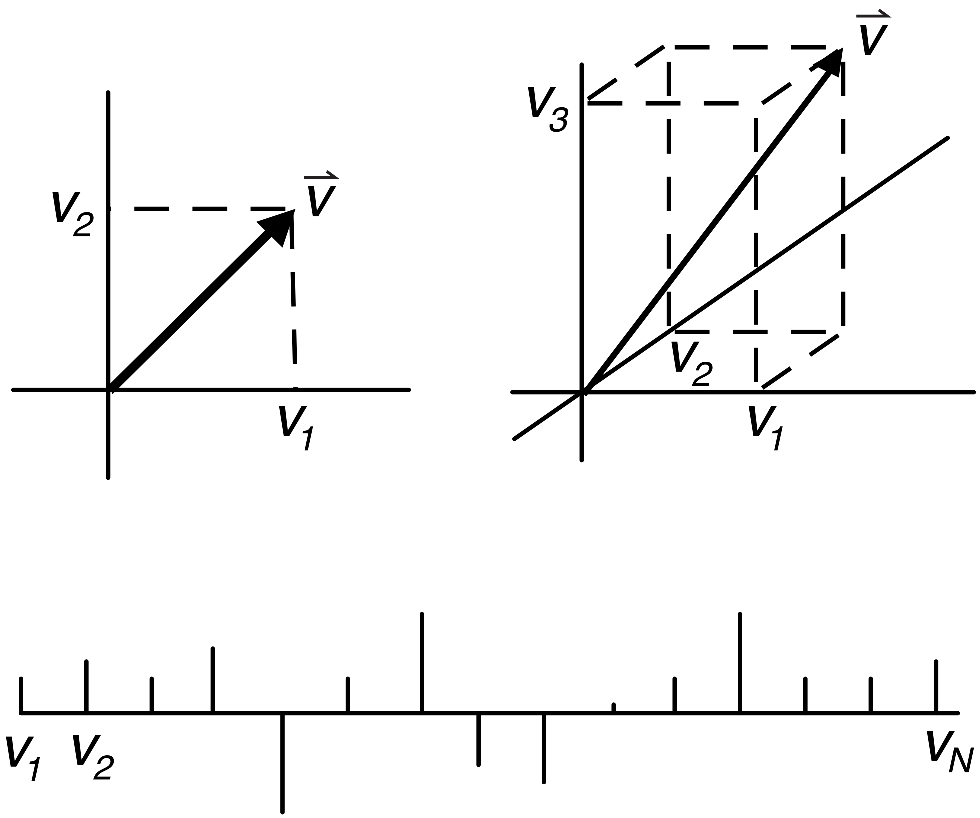
- Glibert Strang, *Linear algebra and its applications*

today's topics

- vectors (geometric picture)
  - vector addition
  - scalar multiplication
- vector norm ("L2 norm")
- unit vectors
- dot product ("inner product")
- linear projection
- orthogonality
- linear combination
- linear independence / dependence

vectors

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix}$$



column vector

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix}$$

in python

```
# make a 3-component column vector
v = np.array([[3], [1], [-7]])
```

transpose

$$\vec{v}^T = \begin{pmatrix} v_1 & v_2 & \cdots & v_N \end{pmatrix}$$

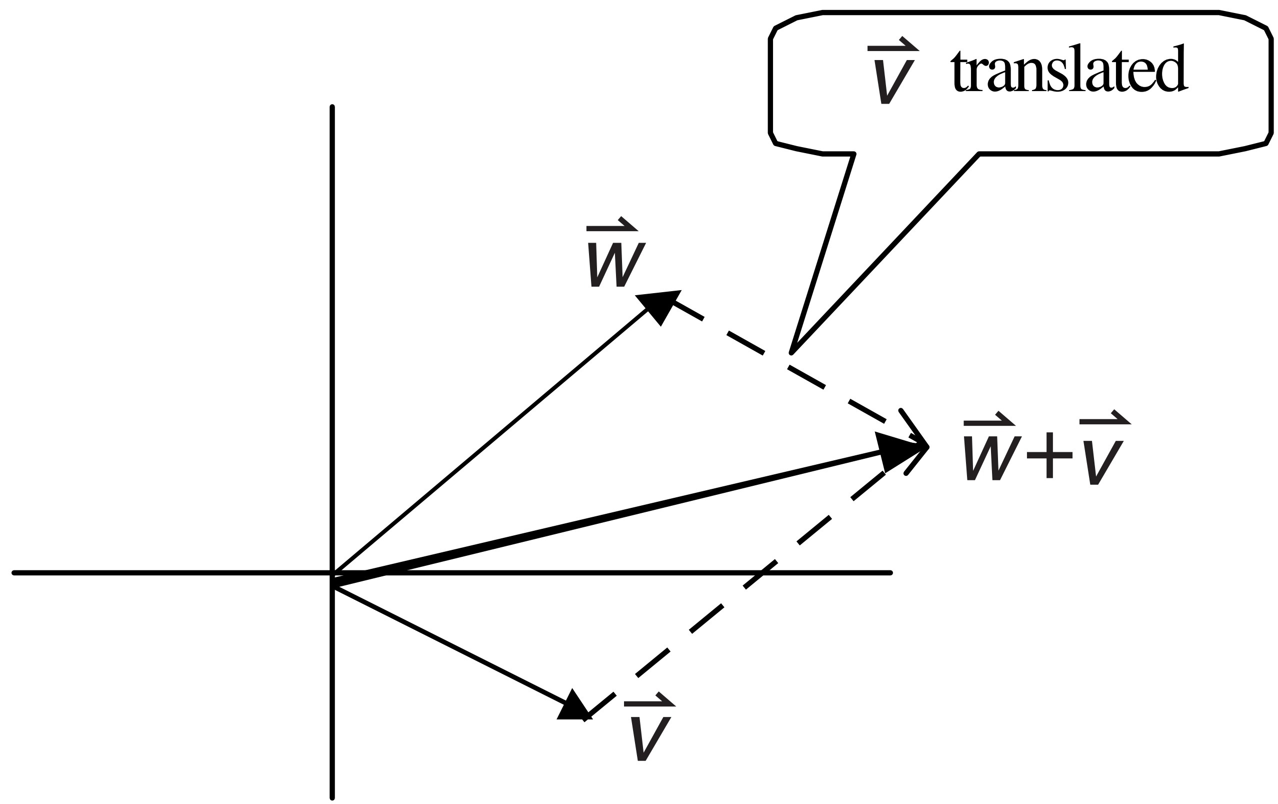
row vector

```
# transpose
v.T
```

```
# create row vector directly
v = np.array([[3,1,-7]]) # row vector
# or
v = np.array([3,1,-7]) # 1D vector
```

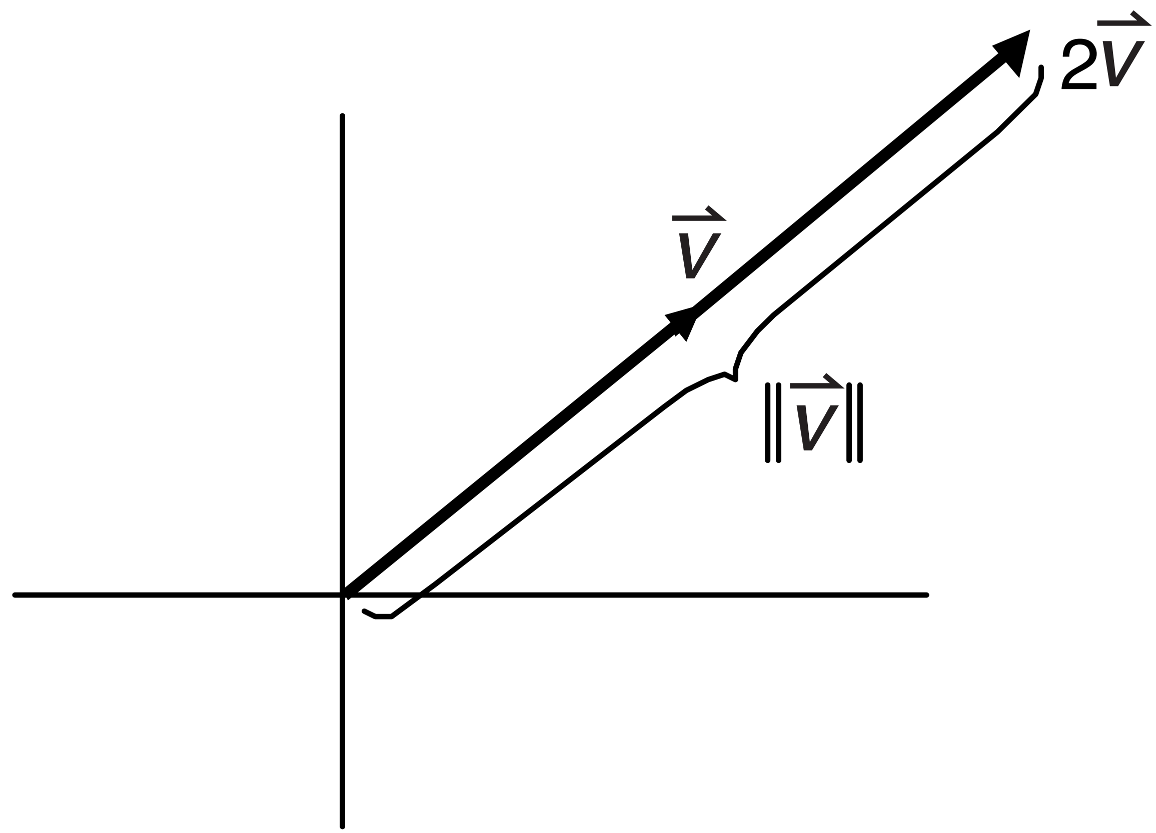
addition of vectors

$$\vec{v} + \vec{w} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix}$$



scalar multiplication

$$a\vec{v} = a\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} av_1 \\ av_2 \end{pmatrix}$$



vector norm (“L2 norm”)

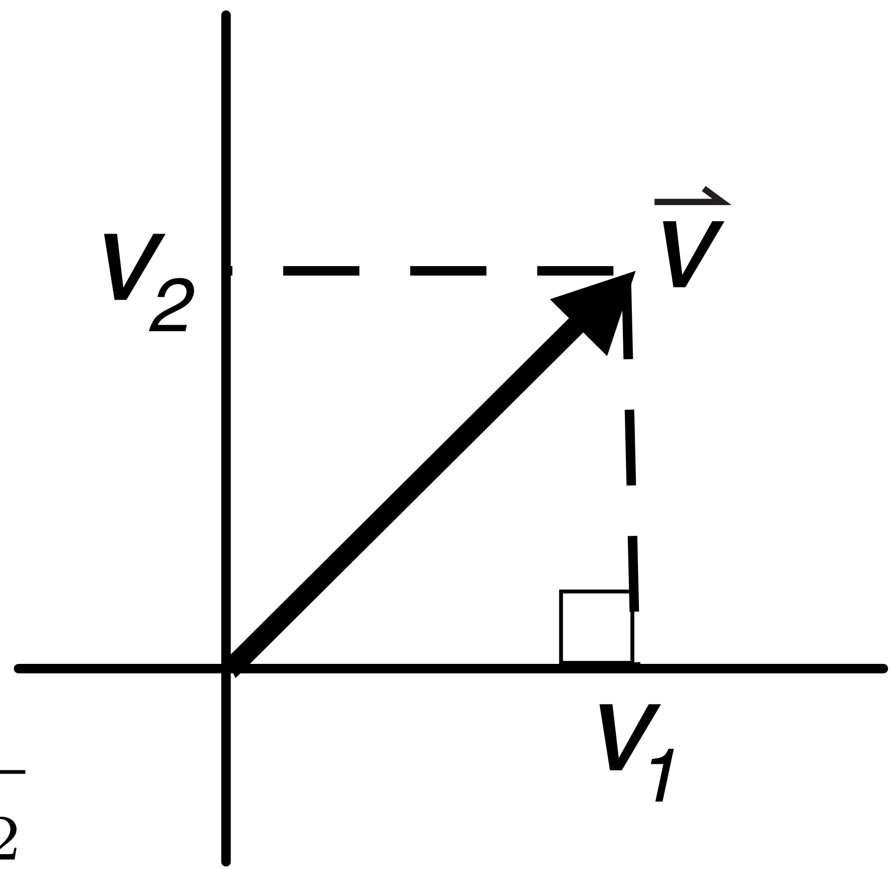
- vector length in Euclidean space

In 2-D:

$$||\vec{v}|| = \sqrt{v_1^2 + v_2^2}$$

In *n*-D:

$$||\vec{v}|| = \sqrt{v_1^2 + \cdots + v_n.^2}$$



vector norm (“L2 norm”)

Exercises compute the vector length (norm of each of the following vectors):

a) [7, 7]
b) [5,5,5,5]
c) [10, 1]
d) [5,5,7]

vector norm (“L2 norm”)

in python

```
# make a vector
v = np.array([1, 7, 3, 0, 1])


# many equivalent ways to compute norm
np.linalg.norm(v)       # built-in function
np.sqrt(np.dot(v,v))    # sqrt of dot product
np.sqrt(v.T @ v)        # sqrt of v-tranpose times v
np.sqrt(sum(v * v))     # sqrt of sum of elementwise product
np.sqrt(sum(v ** 2))    # sqrt of v elementwise-squared


# note use of @ and * and **
# @ - gives matrix multiply
# * - gives elementwise multiply
# ** - gives exponentiation (“raising to a power”)
```

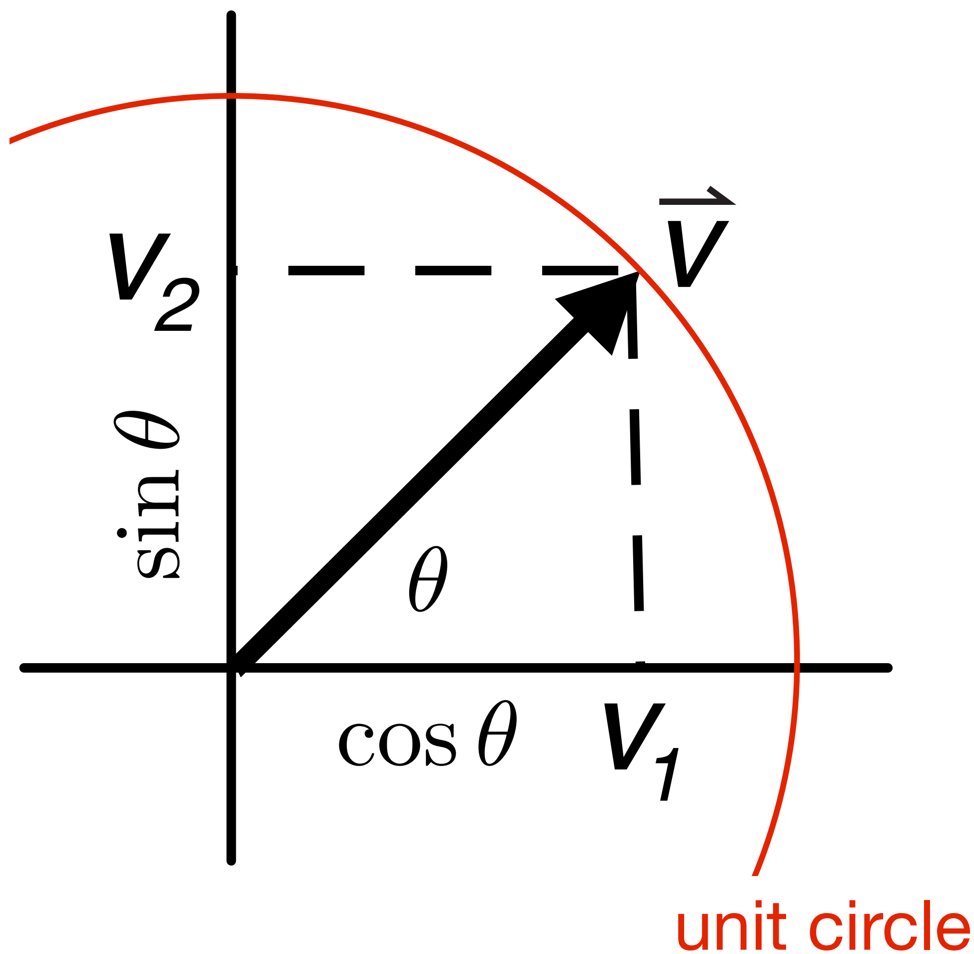
unit vector

- vector such that $||\vec{v}|| = 1$

- <u>in 2 dimensions</u>

$$\vec{v} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

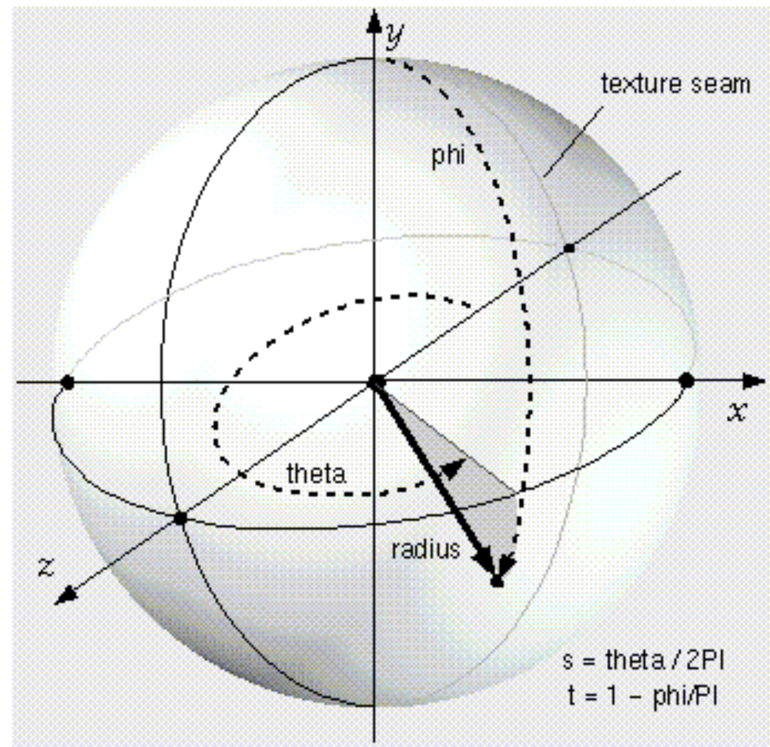
$$\cos^2\theta + \sin^2\theta = 1$$



unit vector

- vector such that $||\vec{v}|| = 1$

- <u>in *n* dimensions</u>



$$v_1^2 + v_2^2 + \ldots + v_n^2 = 1$$

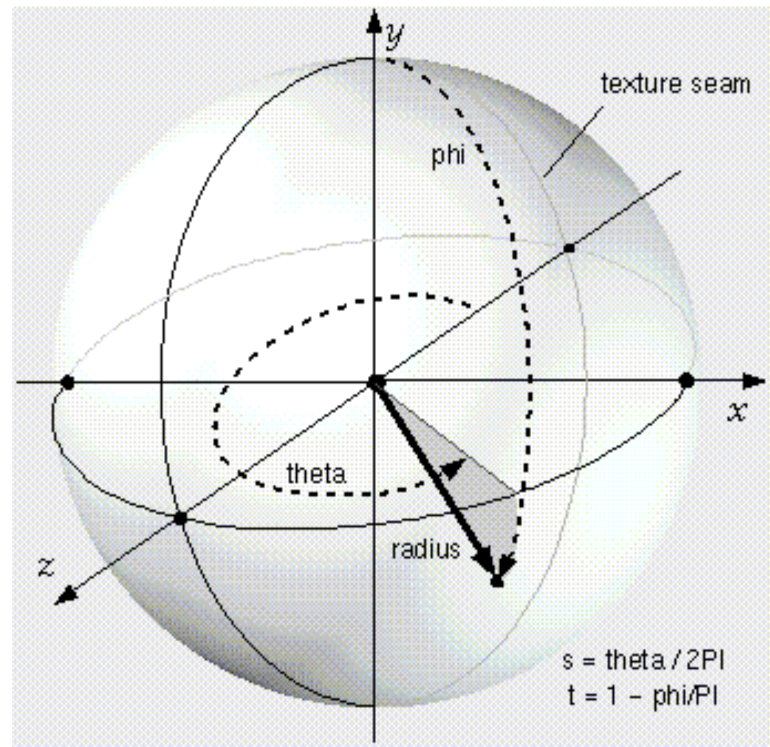
- sits on the surface of an *n*-dimensional hypersphere

unit vector

- vector such that $||\vec{v}|| = 1$

- make any vector into a unit vector via

$$\frac{1}{||\vec{v}||}\vec{v}$$



inner product (aka "dot product")

- produces a scalar from two vectors

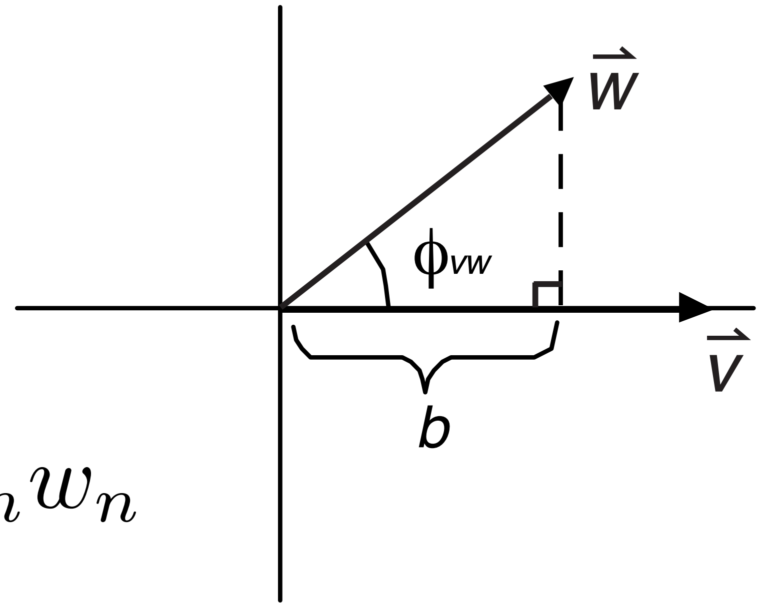
$$\vec{v} \cdot \vec{w}$$

$$\langle \vec{v}, \vec{w} \rangle$$

$$v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$$

$$\|v\| \|w\| \cos \theta$$

$$\vec{v}^T \vec{w} = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$



inner product (aka “dot product”)

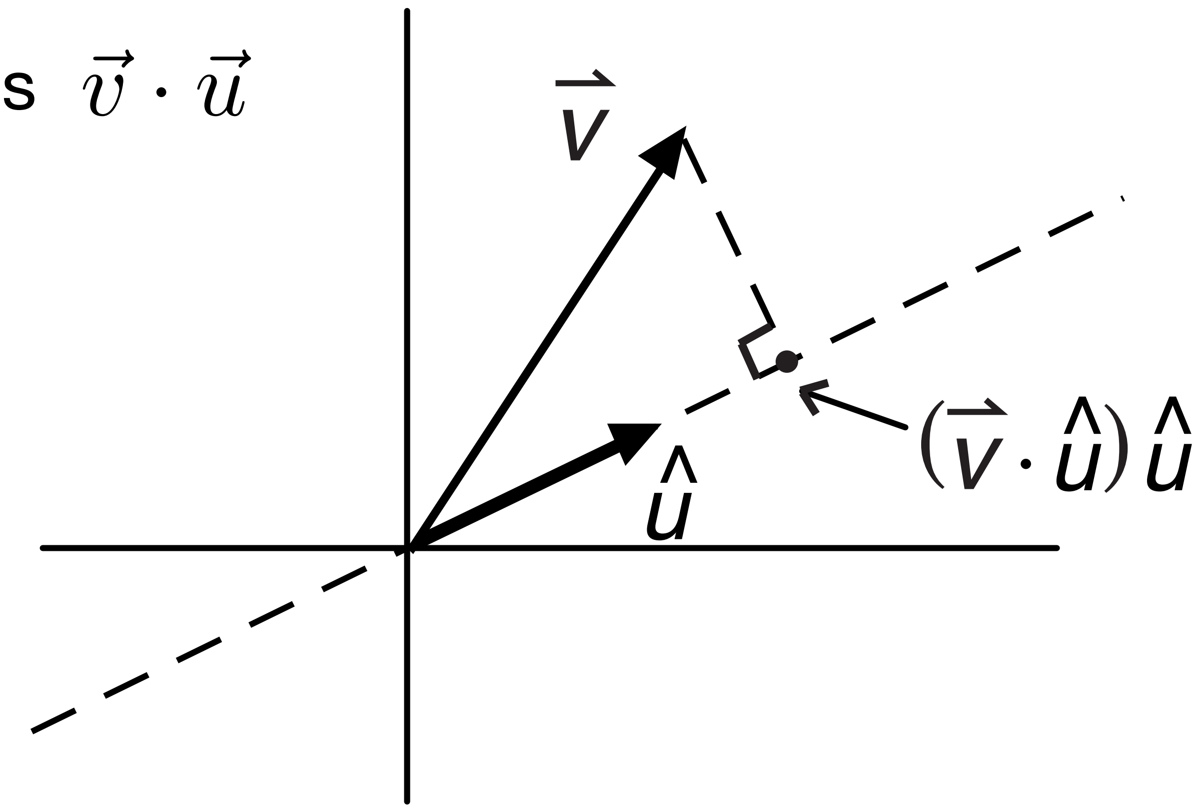
Exercises:

v1 = [1,2,3]
v2 = [3,2,-1]
v3 = [10,0,5]

Compute:
v1•v2, v1•v3, v2•v3

linear projection

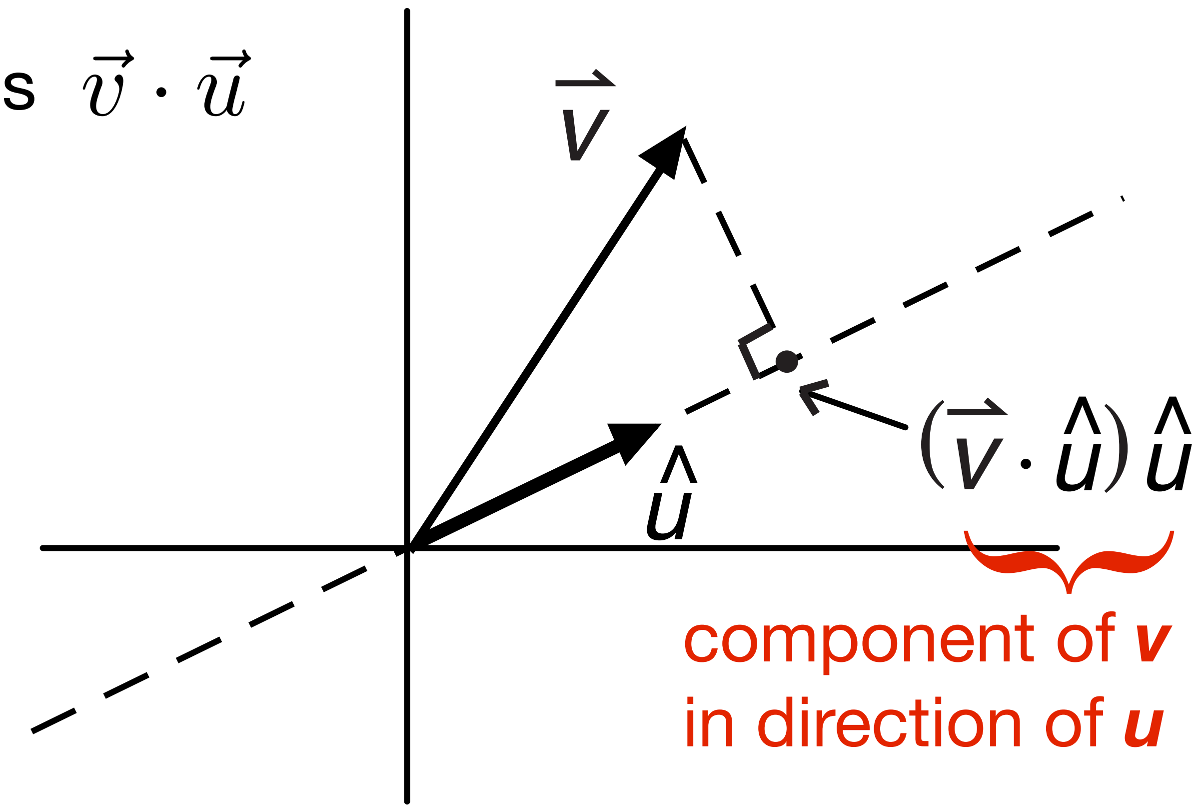
- intuitively, dropping a vector down onto a linear surface at a right angle
- if *u* is a unit vector, length of projection is $\vec{v} \cdot \vec{u}$



- for non-unit vector, length of projection = $\vec{v} \cdot \left(\frac{1}{||\vec{u}||} \vec{u} \right)$

linear projection

- intuitively, dropping a vector down onto a linear surface at a right angle
- if *u* is a unit vector, length of projection is $\vec{v} \cdot \vec{u}$



- for non-unit vector, length of projection = $\vec{v} \cdot \left(\frac{1}{||\vec{u}||} \vec{u} \right)$

Linear Projection Exercise

w = [2,2]
v1 = [2,1]
v2 = [5,0]

Compute:
Linear projection of w onto lines defined by v1 and v2

orthogonality

- two vectors are orthogonal (or “perpendicular”) if their dot product is zero: $\vec{v} \cdot \vec{w} = 0$

orthogonality

- two vectors are orthogonal (or “perpendicular”) if their dot product is zero: $\vec{v} \cdot \vec{w} = 0$