Linear Algebra I:
 vectors and linear projection

NEU 314: Math Tools for Neurosience

Lecture 2 - Tuesday (9/7)

Jonathan Pillow
course website

http://pillowlab.princeton.edu/teaching/mathtools21fall/

1st quiz Thursday

(but no quiz next week)
Linear algebra

“Linear algebra has become as basic and as applicable as calculus, and fortunately it is easier.”

- Glibert Strang, *Linear algebra and its applications*
today’s topics

• vectors (geometric picture)
  • vector addition
  • scalar multiplication
• vector norm (“L2 norm”)
• unit vectors
• dot product (“inner product”)
• linear projection
• orthogonality
• linear combination
• linear independence / dependence
A Geometric Review of Linear Algebra

The following is a compact review of the primary concepts of linear algebra. The order of presentation is unconventional, with emphasis on geometric intuition rather than mathematical formalism. For more thorough coverage, I recommend Linear Algebra and Its Applications by Gilbert Strang, Academic Press, 1980.

Vectors (Finite-Dimensional)

A vector is an ordered collection of $N$ numbers. The numbers are called the components of the vector, and $N$ is the dimensionality of the vector. We typically imagine that these components are arranged vertically (a "column" vector):

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix}$$

Vectors of dimension 2 or 3 can be graphically depicted as arrows. Vectors of higher dimension can be illustrated using a "spike plot".

The norm (or magnitude) of a vector is defined as:

$$||\vec{v}|| = \sqrt{\sum v_n^2}.$$ Geometrically, this corresponds to the length of the vector. A vector containing all zero components has zero norm,

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    v_2 \\
    \vdots \\
    v_N
\end{pmatrix}
\]

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The norm (or magnitude) of a vector is defined as:

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\[\begin{align*}
\text{transpose} & \quad \vec{v}^T = (v_1 \quad v_2 \quad \cdots \quad v_N) \\
\text{row vector} & \quad \text{# create row vector directly} \\
& \quad \text{# or} \\
& \quad \text{v = np.array([[3,1,-7]])} \\
& \quad \text{v = np.array([[3,1,-7]])} \\
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& \quad \text{# 1D vector}
\end{align*}\]
addition of vectors

\[ \vec{v} + \vec{w} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix} \]
scalar multiplication

\[ a\vec{v} = a \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} av_1 \\ av_2 \end{pmatrix} \]
vector norm ("L2 norm")

- vector length in Euclidean space

In 2-D:

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2}$$

In n-D:

$$\|\vec{v}\| = \sqrt{v_1^2 + \cdots + v_n^2}$$
vector norm ("L2 norm")

Exercises compute the vector length (norm of each of the following vectors):

a) [7, 7]
b) [5,5,5,5]
c) [10, 1]
d) [5,5,7]
vector norm ("L2 norm")
in python

# make a vector
v = np.array([1, 7, 3, 0, 1])

# many equivalent ways to compute norm
np.linalg.norm(v)      # built-in function
np.sqrt(np.dot(v,v))   # sqrt of dot product
np.sqrt(v.T @ v)       # sqrt of v-tranpose times v
np.sqrt(sum(v * v))    # sqrt of sum of elementwise product
np.sqrt(sum(v ** 2))   # sqrt of v elementwise-squared

# note use of @ and * and **
# @ - gives matrix multiply
# * - gives elementwise multiply
# ** - gives exponentiation ("raising to a power")
unit vector

- vector such that $||\vec{v}|| = 1$

- in 2 dimensions

$\vec{v} = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$

$\cos^2 \theta + \sin^2 \theta = 1$
unit vector

• vector such that $||\vec{v}|| = 1$

• in $n$ dimensions

$$v_1^2 + v_2^2 + \ldots + v_n^2 = 1$$

• sits on the surface of an $n$-dimensional hypersphere
unit vector

• vector such that $\| \vec{u} \| = 1$

• make any vector into a unit vector via

$$\frac{1}{\| \vec{u} \|} \vec{u}$$
inner product  (aka “dot product”)

• produces a scalar from two vectors

\[ \vec{v} \cdot \vec{w} \]
\[ \langle \vec{v}, \vec{w} \rangle \]
\[ v_1 w_1 + v_2 w_2 + \cdots + v_n w_n \]
\[ \|v\| \|w\| \cos \theta \]
\[ \vec{v}^T \vec{w} = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \]
inner product  (aka “dot product”)

Exercises:

\[ v_1 = [1,2,3] \]
\[ v_2 = [3,2,-1] \]
\[ v_3 = [10,0,5] \]

Compute:
\[ v_1 \cdot v_2, v_1 \cdot v_3, v_2 \cdot v_3 \]
linear projection

- intuitively, dropping a vector down onto a linear surface at a right angle
- if \( u \) is a unit vector, length of projection is \( \vec{v} \cdot \hat{u} \)
- for non-unit vector, length of projection = \( \vec{v} \cdot \left( \frac{1}{||\hat{u}||} \hat{u} \right) \)
linear projection

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component of $\vec{v}$ in direction of $u$
Linear Projection Exercise

\[ w = [2,2] \]
\[ v_1 = [2,1] \]
\[ v_2 = [5,0] \]

Compute:
Linear projection of \( w \) onto lines defined by \( v_1 \) and \( v_2 \)
orthogonality

• two vectors are orthogonal (or “perpendicular”) if their dot product is zero: \( \vec{v} \cdot \vec{w} = 0 \)
orthogonality

- two vectors are orthogonal (or “perpendicular”) if their dot product is zero: \( \vec{v} \cdot \vec{w} = 0 \)

\[ \vec{v} - (\vec{v} \cdot \hat{u}) \hat{u} \]

component of \( \vec{v} \) orthogonal to \( \vec{u} \)

component of \( \vec{v} \) in direction of \( \vec{u} \)