color blindness

• About 8% of male population, 0.5% of female population has some form of color vision deficiency: Color blindness

• Mostly due to missing M or L cones (sex-linked; both cones coded on the X chromosome)
Types of color-blindness:

**dichromat** - only 2 channels of color available (i.e., color vision defined by a 2D subspace) (contrast with “trichromat” = 3 color channels).

Three types, depending on missing cone:

- **Protanopia**: absence of L-cones
- **Deuteranopia**: absence of M-cones
- **Tritanopia**: absence of S-cones

Frequency:

<table>
<thead>
<tr>
<th>Cone Type</th>
<th>M</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Protanopia</td>
<td>2%</td>
<td>0.02%</td>
</tr>
<tr>
<td>Deuteranopia</td>
<td>6%</td>
<td>0.4%</td>
</tr>
<tr>
<td>Tritanopia</td>
<td>0.01%</td>
<td>0.01%</td>
</tr>
</tbody>
</table>

includes true dichromats and color-anomalous trichromats
Scene Viewed by Protanope

Same Scene Viewed by Normal Trichromat
Scene Viewed by Deuteranope

Same Scene Viewed by Normal Trichromat
Scene Viewed by Tritanope

Same Scene Viewed by Normal Trichromat
So don’t call it color *blindness*.

Say: “Hey man, I’m just living in a 2D subspace.”
Other types of color-blindness:

- **Monochromat**: true “color-blindness”; world is black-and-white
- **cone monochromat** - only have one cone type (vision is truly b/w)
- **rod monochromat** - visual in b/w AND severely visually impaired in bright light
Rod monochromacy

Scene Viewed by Rod Monochromat

Same Scene Viewed by Normal Trichromat
Color Vision in Animals

- most mammals (dogs, cats, horses): dichromats
- old world primates (including us): trichromats
- marine mammals: monochromats
- bees: trichromats (but lack “L” cone; ultraviolet instead)
- some birds, reptiles & amphibians: tetrachromats!
Opponent Processes

Afterimages: A visual image seen after a stimulus has been removed

Negative afterimage: An afterimage whose polarity is the opposite of the original stimulus
- Light stimuli produce dark negative afterimages
- Colors are complementary:
  red => green afterimages,
  blue => yellow afterimages
  (and vice-versa)
color after-effects:

lilac chaser:

http://www.michaelbach.de/ot/col-lilacChaser/index.html
last piece: *surface reflectance function*

Describes how much light an object reflects, as a function of wavelength.

Think of this as the *fraction* of the incoming light that is reflected back.
By now we have a complete picture of how color vision works:

**Illuminant**
- defined by its power or “intensity” spectrum
- amount of light energy at each wavelength

**Object**
- defined by its reflectance function
- certain percentage of light at each wavelength is reflected

**Cones**
- defined by absorption spectra
- each cone class adds up light energy according to its absorption spectrum

**Cone responses**
- three spectral measurements
- convey all color information to brain via opponent channels
source (lightbulb) power spectrum

incandescent bulb

object reflectance

× (‘*’ in python)

light from object

“red”

×

“gray”

wavelength (nm)
But in general, this doesn’t happen!
We don’t see a white sheet of paper as reddish under a tungsten light and blueish under a halogen light.

Why?

- **Color constancy**: the tendency of a surface to appear the same color under a wide range of illuminants

- to achieve this, brain tries to “discount” the effects of the illuminant using a variety of tricks (e.g., inferences about shadows, the light source, etc).
Illusion illustrating Color Constancy

(the effects of lighting/shadow can make colors look different that are actually the same!)

Same yellow in both patches

Same gray around yellow in both patches
Exact same light hitting emanating from these two patches

But the brain infers that less light is hitting this patch, due to shadow

CONCLUSION: the lower patch must be reflecting a higher fraction of the incoming light (i.e., it’s brighter)
• Visual system tries to estimate the qualities of the illuminant so it can discount them

• still unknown how the brain does this (believed to be in cortex)
Color vision summary

• light source: defined by illuminant power spectrum

• Trichromatic color vision relies on 3 cones: characterized by absorption spectra (“basis vectors” for color perception)

• Color matching: any 3 lights that span the vector space of the cone absorption spectra can match any color percept

• metamer: two lights that are physically distinct (have different spectra) but give same color percept (have same projection)
  - this is a very important and general concept in perception!

• surface reflectance function: determines reflected light by pointwise multiplication of spectrum of the light source

• adaptation in color space (“after-images”)

• color constancy - full theory of color vision (unfortunately) needs more than linear algebra!
Back to Linear Algebra:

• Orthonormal basis
• Orthogonal matrix
• Rank
• Column / Row Spaces
• Null space
orthonormal basis

• basis composed of orthogonal unit vectors

Working backwards, a set of vectors is said to span a vector space if one can write any vector in the vector space as a linear combination of the set. A spanning set can be redundant: For example, if two of the vectors are identical, or are scaled copies of each other. This redundancy is formalized by defining linear independence. A set of vectors \( \{ \vec{v}_1, \vec{v}_2, ..., \vec{v}_M \} \) is linearly independent if (and only if) the only solution to the equation

\[
\sum_n \alpha_n \vec{v}_n = 0
\]

is \( \alpha_n = 0 \) (for all \( n \)).

A basis for a vector space is a linearly independent spanning set. For example, consider the plane of this page. One vector is not enough to span the plane. Scalar multiples of this vector will trace out a line (which is a subspace), but cannot "get off the line" to cover the rest of the plane. But two vectors are sufficient to span the entire plane.

Bases are not unique: any two vectors will do, as long as they don't lie along the same line. Three vectors are redundant: one can always be written as a linear combination of the other two. In general, the vector space \( \mathbb{R}^N \) requires a basis of size \( N \).

Geometrically, the basis vectors define a set of coordinate axes for the space (although they need not be perpendicular). The standard basis is the set of unit vectors that lie along the axes of the space:

\[
\hat{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \hat{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ldots, \hat{e}_N = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\]

Two different orthonormal bases for the same vector space
Orthogonal matrix

- Square matrix whose columns (and rows) form an orthonormal basis (i.e., are orthogonal unit vectors)

\[ B = \begin{pmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{pmatrix} \]

\[ \vec{b}_i \cdot \vec{b}_i = 1 \]
\[ \vec{b}_i \cdot \vec{b}_j = 0, \; i \neq j \]

Properties:

\[ BB^T = B^T B = I \]
\[ B^{-1} = B^T \]
\[ \| B \vec{v} \| = \| B^T \vec{v} \| = \| \vec{v} \| \]

length-preserving
Orthogonal matrix

- 2D example: rotation matrix

\[
\begin{pmatrix}
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\end{pmatrix}
\]
Rank

• the **rank** of a matrix is equal to
  • # of linearly independent columns
  • # of linearly independent rows

  (remarkably, these are always the same)

**equivalent definition:**

• the rank of a matrix is the *dimensionality* of the vector space spanned by its rows or its columns

**for an \( m \times n \) matrix \( A \): \( \text{rank}(A) \leq \min(m,n) \)**

  (can’t be greater than # of rows or # of columns)
column space of a matrix $W$:

$n \times m$ matrix

\[
W = \begin{pmatrix}
    w_{11} & \cdots & w_{1m} \\
    \vdots & \ddots & \vdots \\
    w_{n1} & \cdots & w_{nm}
\end{pmatrix}
\]

vector space spanned by the columns of $W$

\[
\begin{pmatrix}
    c_1 \\
    \vdots \\
    c_m
\end{pmatrix}
\]

- these vectors live in an n-dimensional space, so the column space is a subspace of $\mathbb{R}^n$
row space of a matrix $W$:

$n \times m$ matrix

$$W = \begin{pmatrix}
  w_{11} & \cdots & w_{1m} \\
  \vdots & & \vdots \\
  w_{n1} & \cdots & w_{nm}
\end{pmatrix}$$

vector space spanned by the rows of $W$

$$\begin{pmatrix}
  \hline
  \hspace{1cm} \quad r_1 \quad \hspace{1cm}
  \hline
  \vdots \\
  \hline
  \hspace{1cm} \quad r_n \quad \hspace{1cm}
  \hline
\end{pmatrix}$$

• these vectors live in an $m$-dimensional space, so the column space is a subspace of $\mathbb{R}^m$
null space of a matrix $W$:

- the vector space consisting of all vectors that are orthogonal to the rows of $W$

$$
\begin{pmatrix}
\ldots & r_1 & \ldots \\
\ldots & \vdots & \ldots \\
\ldots & r_n & \ldots 
\end{pmatrix}
$$

- equivalently: the null space of $W$ is the vector space of all vectors $x$ such that $Wx = 0$.

- the null space is therefore entirely orthogonal to the row space of a matrix. Together, they make up all of $\mathbb{R}^m$. 
null space of a matrix $W$:

$$W = ( \begin{array}{c} v_1 \end{array} )$$

The null space of $W$ is clearly a vector space. Working backwards, a set of vectors is said to span a vector space if one can write any vector in the vector space as a linear combination of the set. A spanning set can be redundant: For example, if two of the vectors are identical, or are scaled copies of each other. This redundancy is formalized by defining linear independence. A set of vectors $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_M\}$ is linearly independent if (and only if) the only solution to the equation

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$$\hat{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ . . . \end{pmatrix}, \hat{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ . . . \end{pmatrix}, ..., \hat{e}_N = \begin{pmatrix} 0 \\ 0 \\ 0 \\ . . . \end{pmatrix}.$$
Change of basis

- Let \( \mathbf{B} \) denote a matrix whose columns form an orthonormal basis for a vector space \( \mathbf{W} \)

\[
\mathbf{B} = \begin{pmatrix}
\vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n
\end{pmatrix}
\]

\[
\vec{b}_i \cdot \vec{b}_i = 1 \\
\vec{b}_i \cdot \vec{b}_j = 0, i \neq j
\]

\[
\mathbf{B}^T \vec{v} = \begin{pmatrix}
\vec{b}_1 \cdot \vec{v} \\
\vdots \\
\vec{b}_n \cdot \vec{v}
\end{pmatrix}
\]

Vector of projections of \( \vec{v} \) along each basis vector