Mathematical Tools for Neuroscience (NEU 314)
Princeton University, Spring 2016
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Lecture 22: Linear Shift-Invariant (LSI) Systems and Convolution
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Linear Shift-Invariant (aka "time-invariant") Systems

An LSI system $f(\vec{x})$ is a system that has two essential properties:

1. Linearity (we know this one already): $f(a\vec{x} + b\vec{y}) = af(\vec{x}) + bf(\vec{y})$, that is, it obeys linear superposition.
2. Shift-invariance: this means that if we shift the input in time (or shift the entries in a vector) then the output is shifted by the same amount. Mathematically, we can say that if $f(\vec{x}(t)) = \vec{y}(t)$, shift invariance means that $f(\vec{x}(t+\tau)) = y(t+\tau)$.

These two properties are independent: e.g., $f(x(t)) = x(t)^2$ is shift-invariant but not linear), and matrix multiplication by an arbitary matrix is linear but (typically) not shift-invariant.

Toeplitz Matrix

Remember that all linear systems can be written in terms of multiplication by a matrix. The special kind of matrix associated with LSI systems is known as a **Toeplitz matrix**, a matrix in which every row is a shifted copy of the one above.

Let's look at an example Toeplitz matrix

$$A = \begin{bmatrix} a & 0 & 0 & 0 \\ b & a & 0 & 0 \\ c & b & a & 0 \\ 0 & c & b & a \end{bmatrix}$$

This clearly corresponds to a LSI system. The response to $x = (1\ 0\ 0\ 0)^\top$ is $(a\ b\ c\ 0)^\top$. If we shift the input by 1, we get input vector $x = (0\ 1\ 0\ 0)^\top$ and shifted output vector $(0\ a\ b\ c)^\top$.

Impulse Response

A linear shift-invariant system can be characterized entirely by its response to an *impulse* (a vector with a single 1 and zeros elsewhere). In the above example, the impulse response was $(a\ b\ c\ 0)$. Note that this corresponds to the pattern found in a single row of the Toeplitz matrix above, but flipped left-to-right.

The impulse response suffices to characterize any response of the LSI system because any input can be written as a linear combination of shifted impulses, so the output is given by the same linear combination of shifted impulse responses.

Convolution

Another way to think about LSI systems is as resulting from a *convolution* between the input and the impulse response. (Often the impulse response is referred to as a "filter" in such settings). If we let $x(t)$ represent an input vector and $a(t)$ represent an impulse response (or filter), then we denote the convolution:

$$\begin{aligned} y(t) &= x(t) * a(t) \\ &= \sum_{s=0}^{t} x(t-s)a(s). \end{aligned}$$

We could of course equally represent the system by

$$\vec{y} = M\vec{x},$$

where M is the Toeplitz matrix with a (flipped, shifted) copy of $\vec{a}$ along each row.

A convolution can also be conceived as "sliding" the filter $a(t)$ along the signal $x(t)$ and taking a dot product with the corresponding portion of $x(t)$ at each location. See figures at end for a picture illustrating this view of convolution.

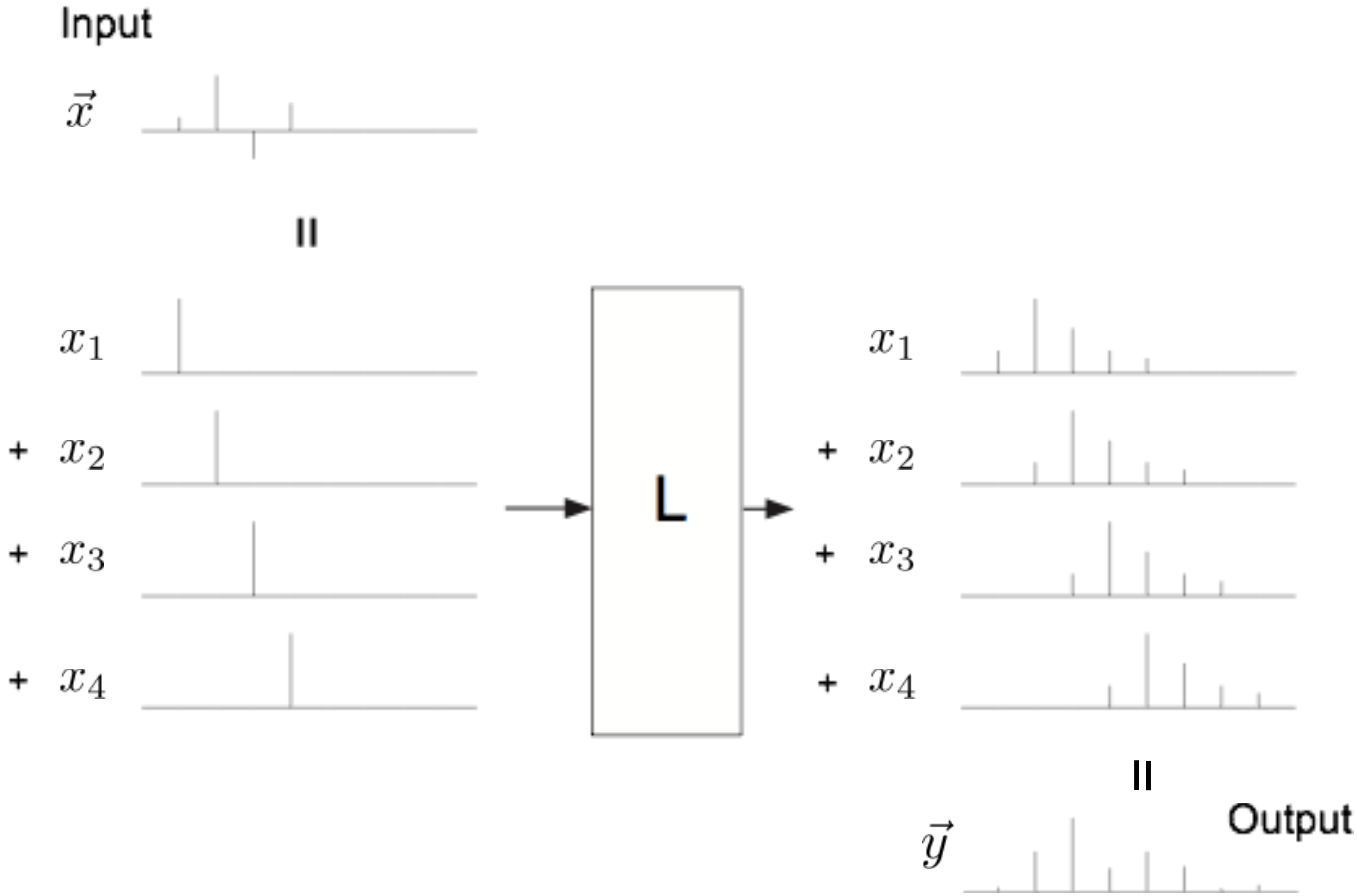
We can also define 2D convolution, which corresponds to shifting an $n \times m$ filter (a matrix) over all locations in a 2D image and taking a dot product.

LSI and Sinusoids

[Quick prelude to next lecture]

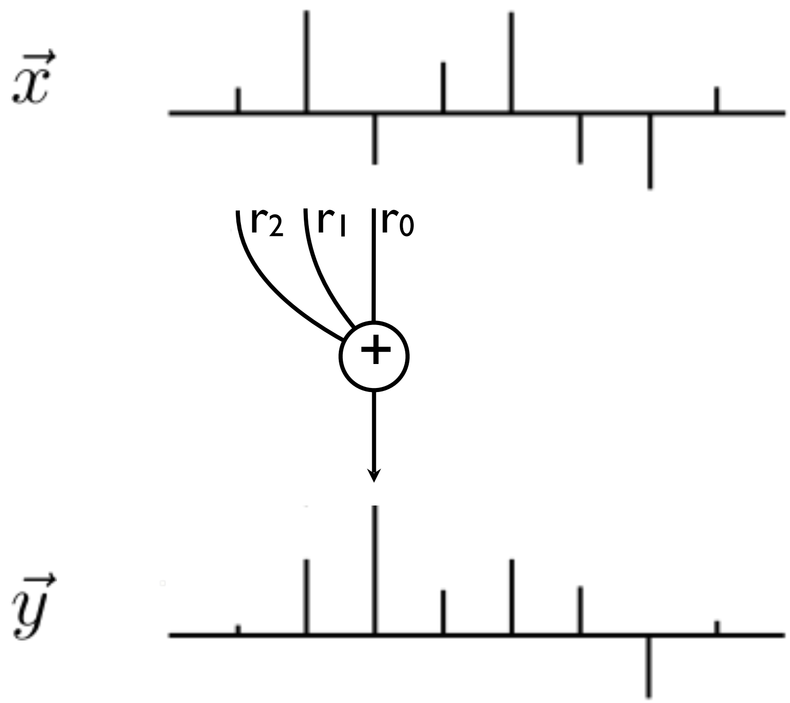
The response of a LSI to a sinusoid is another sinuoid (with the same frequency but altered in amplitude and phase). In other words, sinusoids are the eigenvectors of LSI systems!

LSI system



LSI systems are characterized by their “impulse response”

Convolution



$$y(n) = \sum_m r(n-m)x(m)$$
$$= \sum_m r(m)x(n-m)$$

- Matrix description
- boundaries: zero-padded, reflected, circular
- Examples: impulse, delay, average, difference