Orthogonal matrices, change of basis, rank

Math Tools for Neuroscience (NEU 314)
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Jonathan Pillow
Princeton Neuroscience Institute & Psychology.

Lecture 6
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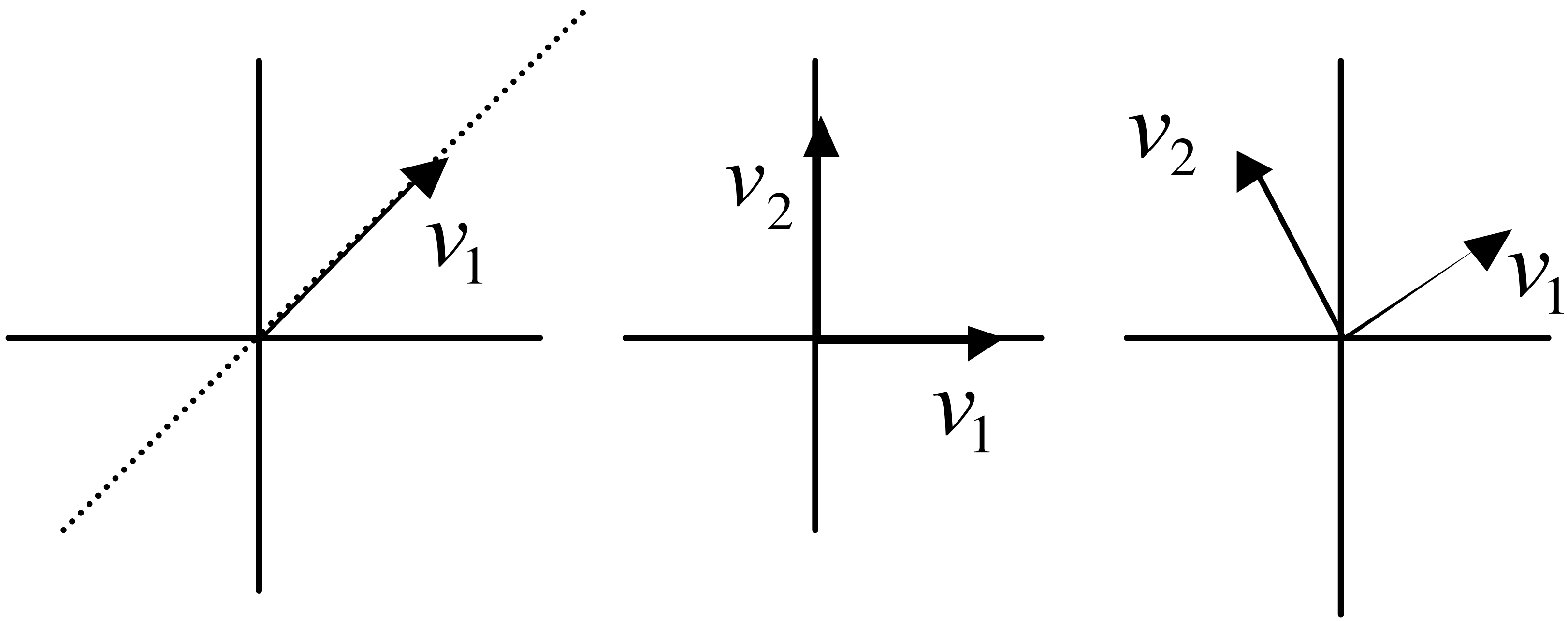
accompanying notes/slides

today's topics

- orthonormal basis
- change of basis
- orthogonal matrix
- rank
- column space and row space
- null space

basis

- set of vectors that can "**span**" (form via linear combination) all points in a vector space



1D vector space (***subspace*** *of R^2*)

Two different (orthonormal) bases for the same 2D vector space

orthonormal basis

- basis composed of orthogonal unit vectors

Change of basis

- Let **B** denote a matrix whose columns form an orthonormal basis for a vector space **W**

$$B = \begin{pmatrix} | & | & & | \\ \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \\ | & | & & | \end{pmatrix}$$

$$\begin{aligned} \vec{b}_i \cdot \vec{b}_i &= 1 \\ \vec{b}_i \cdot \vec{b}_j &= 0, i \neq j \end{aligned}$$

If **B** is full rank (*n x n*), then $BB^T = \left([\vec{b}_i \cdot \vec{b}_j] \right) = I$

$$\implies \vec{v} = BB^T\vec{v}$$

we can get back to the original basis through multiplication by B

Change of basis

- Let **B** denote a matrix whose columns form an orthonormal basis for a vector space **W**

$$B = \begin{pmatrix} | & | & & | \\ \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \\ | & | & & | \end{pmatrix}$$

$$\begin{aligned} \vec{b}_i \cdot \vec{b}_i &= 1 \\ \vec{b}_i \cdot \vec{b}_j &= 0, i \neq j \end{aligned}$$

$$B^T \vec{v} = \begin{pmatrix} \vec{b}_1 \cdot \vec{v} \\ \vdots \\ \vec{b}_n \cdot \vec{v} \end{pmatrix}$$

Vector of projections of ***v*** along each basis vector

Orthogonal matrix

- In this case (full rank, orthogonal columns), **B** is an *orthogonal matrix*

$$B = \begin{pmatrix} | & | & & | \\ \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \\ | & | & & | \end{pmatrix}$$

$$\begin{aligned} \vec{b}_i \cdot \vec{b}_i &= 1 \\ \vec{b}_i \cdot \vec{b}_j &= 0, i \neq j \end{aligned}$$

Properties:

$$BB^T = B^T B = I$$

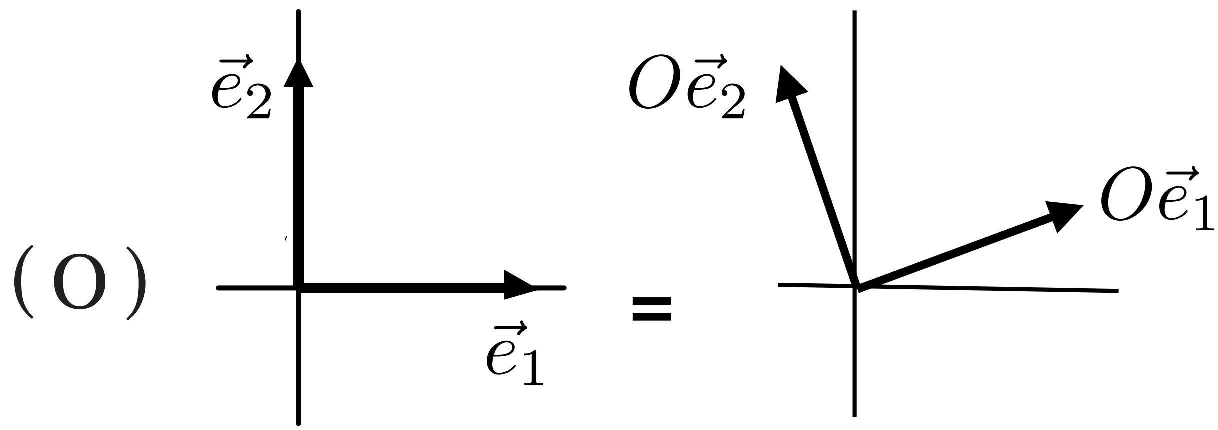
$$B^{-1} = B^T$$

$$||B\vec{v}|| = ||B^T\vec{v}|| = ||\vec{v}||$$

length-preserving

Orthogonal matrix

- 2D example: rotation matrix



$$e.g.\ O = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Rank

- the **rank** of a matrix is equal to
  - # of linearly independent columns
  - # of linearly independent rows

  (remarkably, these are always the same)

equivalent definition:

- the rank of a matrix is the *dimensionality* of the vector space spanned by its rows or its columns

for an *m x n* matrix *A*: $\text{rank}(A) \leq \min(m,n)$

(can’t be greater than # of rows or # of columns)

column space of a matrix W:

$n \times m$ matrix $$W = \begin{pmatrix} w_{11} & \cdots & w_{1m} \\ \vdots & & \vdots \\ w_{n1} & \cdots & w_{nm} \end{pmatrix}$$

vector space spanned by the columns of W $$\begin{pmatrix} | & & | \\ c_1 & \cdots & c_m \\ | & & | \end{pmatrix}$$

- these vectors live in an n-dimensional space, so the column space is a subspace of $\mathbf{R}^n$

row space of a matrix W:

$n \times m$ matrix $$W = \begin{pmatrix} w_{11} & \cdots & w_{1m} \\ \vdots & & \vdots \\ w_{n1} & \cdots & w_{nm} \end{pmatrix}$$

vector space spanned by the *rows* of W

$$\begin{pmatrix} \text{———} \; r_1 \; \text{———} \\ \vdots \\ \text{———} \; r_n \; \text{———} \end{pmatrix}$$

- these vectors live in an m-dimensional space, so the column space is a subspace of $\mathbf{R}^m$

null space of a matrix W:

$n \times m$ matrix

$$\begin{pmatrix} \text{———} & r_1 & \text{———} \\ & \vdots & \\ \text{———} & r_n & \text{———} \end{pmatrix}$$

- the vector space consisting of all vectors that are orthogonal to the *rows* of W

- equivalently: the null space of W is the vector space of all vectors x such that Wx = 0.

- the null space is therefore entirely orthogonal to the row space of a matrix. Together, they make up all of $\mathbf{R}^{m}$.

null space of a matrix W:

$$W = (\text{———} \; v_1 \; \text{———})$$