Trichromatic Theory of Color Vision

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lecture 4.
Linear algebra

“Linear algebra has become as basic and as applicable as calculus, and fortunately it is easier.”

- Glibert Strang, *Linear algebra and its applications*
but first: a few matrix basics

\[ W = \begin{pmatrix} w_{11} & \cdots & w_{1m} \\ \vdots & & \vdots \\ w_{n1} & \cdots & w_{nm} \end{pmatrix} \]

\( n \times m \) matrix

\begin{align*}
\text{can think of it as:} \\
\text{m column vectors} & \quad \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \\
\text{n row vectors} & \quad \begin{pmatrix} \overbrace{\quad \quad \quad \quad \quad}^{r_1} \\ \vdots \\ \overbrace{\quad \quad \quad \quad \quad}^{r_n} \end{pmatrix}
\end{align*}
matrix multiplication: $W \tilde{v}$

One perspective: *dot product with each row of $W$*

\[
\begin{bmatrix}
\vec{u} \\
\end{bmatrix}
\begin{bmatrix}
\vdots \\
\end{bmatrix}
= 
\begin{bmatrix}
\vec{w} \\
\end{bmatrix}
\begin{bmatrix}
\vec{v} \\
\end{bmatrix}
\]

$i^{th}$ component

$i^{th}$ row
matrix multiplication

Other perspective: *linear combination of columns*

\[
\begin{align*}
\begin{pmatrix}
  u_1 \\
  \vdots \\
  u_m
\end{pmatrix}
  \begin{pmatrix}
  \vec{c}_1 \\
  \vdots \\
  \vec{c}_m
\end{pmatrix}
= & \begin{pmatrix}
  v_1 \\
  \vdots \\
  v_m
\end{pmatrix} \\
= & v_1 \cdot \vec{c}_1 + v_2 \cdot \vec{c}_2 + \ldots + v_m \cdot \vec{c}_m
\end{align*}
\]
transpose

• flipping around the diagonal

\[
\begin{pmatrix}
1 & 4 & 7 \\
2 & 5 & 8 \\
3 & 6 & 9
\end{pmatrix}^T = \begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{pmatrix}
\]

square matrix

\[
\begin{pmatrix}
1 & 4 \\
2 & 5 \\
3 & 6
\end{pmatrix}^T = \begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{pmatrix}
\]

non-square

• transpose of a product

\[
(AB)^T = B^T A^T
\]
• If $A$ is a square matrix, its inverse $A^{-1}$ (if it exists) satisfies:

$$AA^{-1} = A^{-1}A = I$$

(eg., for 4 x 4)
(Square) Matrix Equation

\[ A\vec{x} = \vec{b} \]

assume (for now)
square and invertible

left-multiply both sides
by inverse of A:

\[ A^{-1}A\vec{x} = A^{-1}\vec{b} \]

\[ I\vec{x} = A^{-1}\vec{b} \]

\[ \vec{x} = A^{-1}\vec{b} \]
Motivation: James Maxwell’s color-matching experiment

Given any “test” light, you can match it by adjusting the intensities of any three other lights

(2 is not enough; 4 is more than enough & produces non-unique matches)
A little physics background: *light*

- Really: just a particular range of the electromagnetic spectrum
- (We see only the part between 400 and 700 nm)
Q: How many numbers would you need to write down to specify the color of a light source?

Just one?
(“the wavelength”?)
eg. “650”?
Q: How many numbers would you need to write down to specify the color of a light source?

A: It depends on how you “bin” up the spectrum

• One number for each spectral “bin”:

![Energy distribution diagram](image)

example: 13 bins

(a vector!)
Device: **hyper-spectral camera**

- measures amount of energy in each range of wavelengths
- can use thousands of bins (or “frequency bands”), instead of just the 13 shown here
Some terminology for “colored” light:

**spectral** - referring to the wavelength of light

the **illuminant** - light source

**illuminant power spectrum** - this curve.  

(amount of energy (or power) at each frequency)
an illuminant with most power at long wavelengths (i.e., a *reddish* light source)
an illuminant with most power at medium wavelengths (i.e., a greenish light source)
an illuminant with power at all visible wavelengths (a *neutral* light source, or “white light”)

![Diagram showing energy vs. wavelength with a spectrum indicating visible light wavelengths from 400 to 700 nm.](image)
Q: How many measurements of the illuminant spectrum does the human eye take (in bright conditions?)

A: Only 3! One measurement for each cone class.

What mathematical operation do we mean by “measure”?

could also call this axis “absorption” or “sensitivity”
More terminology:

**absorption spectra** - describe response (or “light absorption”) of a photoreceptor as a function of frequency.

The cone absorption spectra are **basis vectors** for a three-dimensional **vector space** (a **subspace**) within the space of all spectra.
Written in a linear algebra setting

\[ \vec{y} = M \vec{x} \]

cone responses

cone absorption spectra

illuminant spectrum
So: single cone can’t tell you anything about the color of light!
(“It’s just giving you dot products, man!”)

Colored stimulus

Response of your “S” cones
Color vision

Our color vision relies on comparing the responses of three cone classes

![Color vision diagram](image)
cone responses: 40 175 240
(dot products)

Metamers
- Illuminants that are physically distinct but perceptually indistinguishable
Two lights $x_1$ and $x_2$ “match” iff

$$M\vec{x}_1 = M\vec{x}_2$$

(i.e., they evoke the same cone responses)

So in linear algebra terms: metamers refer to an entire (affine) subspace of lights that have the same linear projection onto the 3 cone absorption spectra basis vectors.
James Maxwell (1831–1879): color-matching experiment

- Any “test” light (“vector”), can be matched by adjusting the intensities of any three other lights (“basis vectors”)
- 2 is not enough; 4 is more than enough
Implication: tons of things in the natural world have different spectral properties, but look the same to us.

But, great news for the makers of TVs and Monitors: any three lights can be combined to approximate any color.

Single-frequency spectra produced by (hypothetical) monitor phosphors produce “metameric match” to illuminant #1 (or any other possible illuminant).
Close-up of computer monitor, showing three phosphors, (which can approximate any light color)
Written as a linear algebra problem

\[ \vec{x} = P \vec{z} \]

- \( \vec{x} \): spectrum produced
- \( P \): spectra of monitor phosphors
- \( \vec{z} \): input to each phosphor

Diagram notes:
- \( p_3, p_2, p_1 \): elements of the matrix
- "input to each phosphor"