Mathematical Tools for Neuroscience (NEU 314)
Princeton University, Spring 2016
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Homework 5: Probability Basics

Due: Tuesday, April 5, 9:59am

Marginal & Conditional Probability Densitites

Load the file pdfData2D.mat, which contains a discretely sampled 2D probability density. The variables defined include:

x	=	vector of x points
y	=	vector of y points
Pxy	=	2D matrix, whose i,j'th entry is the probability $P(y = \mathtt{y}(i), x = \mathtt{x}(j))$

(Note that sum(Pxy(:))*dx*dy = 1, where dx = x(2)-x(1) = .1 = dy. You can visualize this density in matlab via the command: imagesc(x,y,Pxy); axis image; axis xy;

1. From this joint two-dimensional density, compute and make plots (or images) of

   - $P(x)$ - the marginal distribution over x
   - $P(y)$ - the marginal distribution over y
   - $P(y|x = 5)$ - the conditional over y given $x = 5$.
   - $P(x|y)$ - the full image of the conditional density $P(x|y)$.
   - $P(y)P(x)$ - image of the independent approximation to $P(x, y)$.

   (one point each).

2. Three common statistics one might wish to compute from a density are its mean, mode, and median.

   - The *mean* is the average value, given by $\mathbb{E}[x] = \int xP(x)dx$ when $P(x)$ is a pdf , and $\mathbb{E}[x] = \sum_i x_i P(x_i)$ when $P(x)$ is a pmf.
   - The *mode* is the value x where $P(x)$ takes its maximum. We can write this (fancily, if we like) as $\arg\max_x P(x)$.
   - The *median* is the value of x where half the probability mass $P(x)$ is to the left (smaller than t) and half the probability is to the right (greater than t). In math notation, this corresponds to saying that the mode t satisfies

$$\int_{-\infty}^{t} P(x)\, dx = \int_{t}^{\infty} P(x)\, dx = \frac{1}{2}$$

Compute the mean, mode and median of the marginal $P(x)$ and of the conditional $P(x|y=3)$. (6 points total).

Bayes' Rule

3. Take $P(y|x)$ as computed above, and assume that it describes the noisy process by which a variable x is transformed into a noisy measurement y. Suppose that we are given a prior distribution over x: $P(x) = \mathcal{N}(5, 4)$, a Gaussian with mean 5 and variance 4 (stdev = 2). Compute the posterior distribution $P(x|y)$ given this new prior, and plot an image of it (i.e. showing $P(x|y)$ for every possible value of x and y).

4. Make a plot showing the *maximum a posteriori* (MAP) estimate x_{MAP} as a function of noisy measurement y, where $x_{MAP} = \arg\max_x P(x|y)$ is the mode of the posterior distribution. That is, plot the most probabile value of x under the posterior, for every value of y.

Multivariate Gaussians

5. Write your own function `mvnDensity`.m, which evaluates a multivariate Gaussian density at a single point given a mean vector and a covariance matrix. (No cheating by using `normpdf` or `mvnpdf`!). (2 points). You may recall (from the notes) that the formula for a multivariate Gaussian density is:

$$P(x) = \frac{1}{\sqrt{|2\pi\Lambda|}} \exp\left(-\frac{1}{2}(\vec{x}-\vec{\mu})^\top \Lambda^{-1}(\vec{x}-\vec{\mu})\right),$$

where $|\cdot|$ denotes the matrix determinant. The function should take as input a vector $\vec{x}$, a vector mean $\vec{\mu}$ and the covariance matrix Λ. (2 points).

6. Use `meshgrid` to create a lattice of X and Y points, and then generate a 2D ("bivariate") Gaussian density over this lattice. Make images (use `imagesc` in matlab) showing a bivariate Gaussian with positive correlation, and another with negative correlation. (1 point each).

7. Write another function, `mvnRand`.m, that draws samples from a multivariate Gaussian density given a mean μ and covariance matrix Λ. Recall that we can generate Gaussian samples with covariance AA^T and mean μ using the affine mapping $A\vec{x} + \vec{\mu}$, where $\vec{x}$ is sampled from a Gaussian with zero-mean and identity covariance. (Thus, if Λ is the desired covariance matrix, we need A such that $AA^T = \Lambda$. (**Hint**: how can you use the SVD of Λ to find a suitable A?). (2 points).

8. Generate two scatter plots showing 100 samples from each of the two 2D Gaussians whose pdfs you visualized above. (1 point each).