Mathematical Tools for Neuroscience (NEU 314)
Princeton University, Spring 2016
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Homework 1: linear algebra & trichromacy

Due: Tuesday, Feb 23, 9:59am

Formatting Instructions:

- Please prepare your homework submission as a single script (m-file), with %% separating each problem or sub-problem, so that each can be executed with a `ctrl-enter`. See the posted solution for homework 0 to see an example of the desired formatting.
- Use comments (lines preceded by `%`) to respond to any questions requiring additional explanation.
- Several problems will require writing a separate function (which will be called from your script). Place the main script and any necessary functions in a folder called `lastname_hw1`. Zip this folder and email it to the TA for your section by the due date.
- Problems or lettered sub-problems are worth 1 point each, unless otherwise specified.
- Late submissions will be penalized one letter grade (10%) per day.

Linear Algebra

1. Linear Projection.

   a. Create two random 10-component column vectors $\vec{x}$ and $\vec{u}$ using `randn`, and make $\vec{u}$ into a unit vector.

   b. Compute the linear projection of $\vec{x}$ onto $\vec{u}$, that is, the component of $\vec{x}$ that lies in the 1D subspace defined by $\vec{u}$. Call it `x_proj`.

   c. Compute the component of $\vec{x}$ that is orthogonal to $\vec{u}$. Call it `x_orth`.

   d. Are `x_proj` and `x_orth` orthogonal? Give evidence for why or why not.

   e. Show that the original $\vec{x}$ is equal to the sum of its projection onto $\vec{u}$ and `x_orth`.

2. Linear independence.

   a. Are the vectors $[6; -2]$ and $[-3; 1]$ linearly dependent or linearly independent? (Why or why not?).

   b. What about the vectors $[6; -2; 7]$ and $[-3; 1; 7]$? Linearly dependent or linearly independent?

   c. Write a Matlab function `checkIfIndependent.m` that takes in a pair of column vectors $\vec{u}$ and $\vec{v}$ as input and returns `1` if they are linearly independent and returns `0` if they are linearly dependent. Check that it works correctly when called with the vectors given in *a* and *b* above. (2 points).

3. *Gram-Schmidt orthogonalization* (8 points). Gram-Schmidt orthogonalization is a classic method for constructing an orthonormal basis from an arbitrary set of n linearly independent vectors. This method proceeds as follows. For the first vector, rescale it to be a unit vector. For each subsequent vector, orthogonalize it with respect to each basis vector already created (i.e., subtract off the component pointing in the direction of each already-created basis vector), and then rescale it to be a unit vector.

   Write your own function (`gramSchmidt.m`) to perform Gram-Schmidt orthogonalization. It should take an arbitrary $n \times m$ matrix as input, and return a matrix whose columns form an orthonormal basis for the same subspace spanned by the columns of the matrix passed in. If the input vectors are linearly dependent (which will occur if you get a zero vector after orthogonalizing a vector with respect to the previous basis vectors), your function should output an error (e.g., "`error('columns are linearly dependent!');`").

   Call your function with a random 3×2 matrix, a random 3×3 matrix, and a random 3×4 matrix. Check that in the first two cases, the columns of the matrix `M` passed back are unit vectors, and are orthogonal to each other.

   What do you get if you matrix multiply the transpose of `M` times `M`?

Color Vision

Download the file `colrmatch.mat` from piazza or the course website, and load it in your MATLAB environment (Copy the file into your working directory and type "`load colrmatch`") . This file contains a number of matrices and vectors related to the color matching problems discussed in class. The variable `Phosphors` is an $N \times 3$ matrix containing the emission spectra (i.e., illuminant power spectrum) of three standard color monitor phosphors. Each column of `Phosphors` is a vector describing the illumination spectrum for the corresponding phosphor. The variable `M` is a $3 \times N$ "color-matching matrix" that can be used to determine if an observer will perceive two lights as having the same color. Specifically, two lights, defined by illuminant power spectrum vectors $\vec{l}_1$ and $\vec{l}_2$, are said to match in color if and only if $M\vec{l}_1 = M\vec{l}_2$. For these problems $N = 31$, corresponding to a sampling of the wavelength spectrum from 400nm to 700nm in increments of 10nm. The vector of wavelengths is given in the variable `wavelengths`.

4. *a*. Create a vector called `light` and set it to be a random vector of length 31, with all positive components. We will consider the components of this vector to be samples of the wavelength spectrum of a particular light. Compute a length-3 vector `x` containing the settings of the three monitor phosphors in `Phosphors` that will match the appearance of this randomly chosen test light? (You might want to start with paper and pencil before trying to solve this in Matlab). (2 points).

   b. Compute the (N-dimensional) power spectrum of this matching light by multiplying `Phosphors` by `x`. Verify that it satisfies the "matching" criterion described above. Now plot it together with the test light `light`, and explain why the two lights look so different. (2 points).

   Bonus: what additional constraints on `x` would we need to take into account if considering the problem of matching an arbitrary test light on a real monitor? (1 point).

5. The variable `Cones` contains approximate spectral sensitivities of the three color photoreceptors (cones) in the human eye. They are ordered such that the first row (`Cones`(1,:)) corresponds to the L (long-wavelength, or red) cones, the second row (`Cones`(2,:)) the M (green) cones, and the third row (`Cones`(3,:)) the S (blue) cones. Plot these using `plot`, using the variable `wavelengths` for the x values. Add axis labels and a legend. (1 point).

   Verify that the three cones provide a physiological explanation for the matching experiment carried out by Maxwell. Specifically, give an informal proof (write the linear algebra, and then test it in MATLAB) of the fact that the cone absorptions are equal for any pair of lights that match (i.e. any pair that produce the same knob settings in the matching experiment). [Hint: What is the relationship between the vector space spanned by the rows of `M` and the vector space spanned by the rows of `Cones`?) (4 points).